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# Stabilization of Continuous-Time Markov Jump Linear Systems with Defective Statistics of Modes Transitions ⋆

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**Abstract:** This paper concerns the stabilization problem of a class of Markov jump linear system (MJLS) with defective statistics of modes transitions in the continuous-time domain. Differing from the recent separate studies on the so-called uncertain transition probabilities (TPs) and partially unknown TPs, the defective statistics about modes transitions in this study take the two situations into account in a composite way. The scenario is more practicable in that it divides the TPs into three sets: known, uncertain and unknown. The necessary and sufficient conditions for the stability and stabilization of the underlying system are obtained by fully using the properties of the transition rate matrix (TRM) and the convexity of uncertain domains. The monotonicity, in concern of the existence of the admissible stabilizing controller, is observed when the unknown elements become uncertain and the intervals of the uncertain ones become tighter. Numerical examples are provided to verify the theoretical findings.

*Keywords:* Markov jump linear system, Transition probability, Defective statistics.

## 1. INTRODUCTION

As a class of stochastic hybrid systems, Markov jump linear system (MJLS) has been extensively studied in control discipline over past decades, e.g., Boukas [2005], Costa et al. [2005]. The systems contain a finite number of modes which jump one another, and the jumps are determined by a transition rate matrix (TRM) in continuous-time domain or transition probability matrix (TPM) in discrete-time domain. The knowledge of TRMs or TPMs, as the statistics of the modes transitions, are commonly assumed to be available *a priori*. In recent years, to relax this ideal assumption, increasing attention has been devoted to the defective statistics about the modes transitions. Two concepts in the studies have been proposed, the so-called uncertain transition probabilities (TPs) and partially unknown TPs, see for example, De Souza et al. [2006], Karan et al. [2006], Xiong et al. [2005], and Wang et al. [2010], Zhang and Boukas [2009], Zhang et al. [2008], respectively.

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In the context of uncertain TPs, the elements in a TRM and TPM are considered to be uncertain within an interval and two description ways are adopted, the norm-bounded and the polytopic uncertainty description. Correspondingly, the true elements in a TRM or a TPM are unknown but belong to a *given* range with lower and upper bounds, e.g., Xiong et al. [2005], or a *given* polytope with a certain number of vertices, e.g., De Souza et al. [2006]. It should be noted that such *given* information are assumed to be provided in practical samplings and computations of obtaining the statistics of the transitions. On the other hand, the concept of partially unknown TPs assume that some elements in a TRM or TPM are exactly known, and others are not and also without any further *given* information of the statistics, e.g., Zhang and Boukas [2009]. Therefore, it can be well understood that the concept of partially unknown TPs is more general and the concept of uncertain TPs is less conservative since more information are "contrived" there.

In fact, the unknown elements in the context of partially unknown TPs can be also viewed as uncertain within their natural intervals (either norm-bounded or polytope uncertainty description), which can be calculated by the known elements and the properties that the sum of each row is zero in a TRM or one in a TPM. Therefore, the two concepts of the defective statistics are mathematically interrelated. Nevertheless, from different viewpoints, they lead to the studies on the underlying MJLSs with different levels of available knowledge being reflected. Note that, so far, these two lines of attacks to the defective statistics of modes transitions are still separate. However, a more practicable scenario that designers may encounter is that some elements are exactly known, some elements are unknown and others are unknown but with tighter intervals (not natural intervals) offered from statistics. It also means that in the context of partially unknown TPs, a part of unknown elements further become uncertain ones with extra information contrived by either norm-bounded or polytopic uncertainty description. The goal should be sought for in practice, under the circumstance that the exactly known TPs can not be obtained.

In this paper, the defective statistics about modes transitions take the uncertain TPs and partially unknown TPs descriptions into account in a composite way. The case is more practicable, where the elements in a TRM or a TPM are divided into three sets: known, uncertain and unknown. The necessary and sufficient conditions for the stability and stabilization of the underlying system in continuous-time domain are obtained by fully using the properties of the TRM and the convexity of uncertain domains. The monotonicity, in concern of the existence of the admissible stabilizing controller, is observed when the unknown elements become uncertain or the intervals of the uncertain ones become tighter. The remainder of the paper is organized as follows. Section II gives the problem formulation. Section III is devoted to the main results of the paper. The theoretical findings are verified via numerical examples in Section IV and the paper is concluded in Section V.

*Notation:* The notations used in this paper are fairly standard, and can be found in the relevant literature of MJLSs. We omit them here due to the space limit.

## 2. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following continuous-time Markov jump linear systems (MJLSs), defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$:

$$\dot{x}(t) = A(r_t)x(t) + B(r_t)u(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^l$ is the control input. The Markov stochastic process $\{r_t, t \geq 0\}$, taking values in a finite set $\mathcal{I} \triangleq \{1, ..., N\}$, governs the switching among the different system modes and has the following mode transition probabilities:

$$\Pr(r_{t+h} = j | r_t = i) = \begin{cases} \lambda_{ij}h + o(h), & \text{if } j \neq i \\ 1 + \lambda_{ii}h + o(h), & \text{if } j = i \end{cases}$$

where $h > 0$, $\lim_{h\to 0}(o(h)/h) = 0$ and $\lambda_{ij} \geq 0$ $(i, j \in \mathcal{I}, j \neq i)$ stands for the switching rate from mode $i$ at time $t$ to mode $j$ at time $t + h$, and $\lambda_{ii} = -\sum_{j=1,j\neq i} \lambda_{ij}$ for all $i \in \mathcal{I}$. Thus, the Markov process transition rate matrix $\mathbf{\Lambda}$ is defined by:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2N} \\ & & \ddots & \\ \lambda_{N1} & \lambda_{N2} & \cdots & \lambda_{NN} \end{bmatrix}$$

The set $\mathcal{I}$ contains $N$ modes of the system (1) and for $r_t = i \in \mathcal{I}$, the system matrices of the $i^{th}$ mode are denoted by $(A_i, B_i)$, which are real known with compatible dimensions.

In this paper, the statistics about the transition rates of the jumping process is considered to be defective. Specifically, we assume that some elements in matrix $\mathbf{\Lambda}$ are not exactly known. They may be uncertain within given intervals offered from statistics, or they do not have such available intervals. We coin the former as "uncertain" elements, and the latter as "unknown" ones here. Then, take a MJLS with 5 operation modes for example, the transition rates matrix $\mathbf{\Lambda}$ may be as:

$$\begin{bmatrix} \lambda_{11} & \hat{\lambda}_{12} & [\overline{\lambda}_{13}\ \underline{\lambda}_{13}] & \hat{\lambda}_{14} & [\overline{\lambda}_{15}\ \underline{\lambda}_{15}] \\ \hat{\lambda}_{21} & \hat{\lambda}_{ij} & [\overline{\lambda}_{23}\ \underline{\lambda}_{23}] & \lambda_{24} & [\overline{\lambda}_{25}\ \underline{\lambda}_{25}] \\ \hat{\lambda}_{31} & \hat{\lambda}_{32} & [\overline{\lambda}_{33}\ \underline{\lambda}_{33}] & \hat{\lambda}_{34} & \lambda_{35} \\ \hat{\lambda}_{41} & \hat{\lambda}_{42} & \hat{\lambda}_{43} & \hat{\lambda}_{44} & \lambda_{45} \\ [\overline{\lambda}_{51}\ \underline{\lambda}_{51}] & \hat{\lambda}_{52} & \lambda_{53} & \hat{\lambda}_{54} & [\overline{\lambda}_{55}\ \underline{\lambda}_{55}] \end{bmatrix} \tag{2}$$

where $\forall i \times j \in \mathcal{I} \in \mathcal{I}$, each unknown element is labeled with a hat " $\hat{}$ ", and each uncertain element is denoted by a range with lower and upper bounds $\overline{\lambda}_{ij}$ and $\underline{\lambda}_{ij}$.

*Remark 1.* Note that for the unknown elements that we named, they actually have their "natural intervals" which can be readily calculated by the exactly known elements, the intervals of uncertain elements and the property that the sum of all the elements in each row of a TRM is zero. The reason we sort the uncertain and unknown elements here is that, although we cannot know some element exactly, we may obtain a tighter interval for it, not only its natural one. Therefore, the uncertain elements relatively have more statistics knowledge contained in their updated intervals.

*Remark 2.* Note also that it is intuitionistic to represent the uncertain element by a range as shown in (2). However, the ranges provided in statistics may be invalid in ensuring the property of TRM. In addition, the interactive ranges of more than two uncertain elements may lead to a situation that the upper bounds of some intervals cannot be reached simultaneously. The two cases are shown in Table 1, respectively, taking the $1^{st}$ row in (2) for example. It can be seen that the description about the uncertain elements by means of ranges with lower and upper bounds need be adjusted to satisfy the property of the TRM.

| | the $1^{st}$ row in TRM (2) |
|---|---|
| Case I | $[-0.7\ \hat{\lambda}_{12}\ [0.4\ 0.5]\ \hat{\lambda}_{14}\ [0.4\ 0.6]]$ |
| Case II | $[-1.1\ \hat{\lambda}_{12}\ [0.4\ 0.5]\ \hat{\lambda}_{14}\ [0.4\ 0.7]]$ |

Table 1. Two invalid cases of the uncertain elements

Here, for the tractability of analysis and synthesis of the underlying MJLS, we change the description of the uncertain elements in $\mathbf{\Lambda}$ by ranges as shown in (2) to the description by polytopic uncertainties as below, namely, the uncertain elements in $\mathbf{\Lambda}$ belong to a polytope $\mathbf{P}_\Lambda$ with vertices $\mathbf{\Lambda}_r, \forall r = 1, 2, \cdots, M$, i.e.,

$$\mathbf{P}_\Lambda \triangleq \left\{ \mathbf{\Lambda} | \mathbf{\Lambda} = \sum_{r=1}^{M} \alpha_r \mathbf{\Lambda}_r; \alpha_r \geq 0, \sum_{r=1}^{M} \alpha_r = 1 \right\} \tag{3}$$

where $\mathbf{\Lambda}_r, r = 1, 2, \cdots, M$, are given TRMs still containing partially unknown elements. The means of description for uncertain TPs can be referred to De Souza et al. [2006] for more details. It is worth emphasizing that in (3), the property of each TRM $\mathbf{\Lambda}_i$ holds and the property of TRM $\mathbf{\Lambda}$ will be accordingly satisfied.

For simplicity, $\forall i \in \mathcal{I}$, we denote $\mathcal{I} = \mathcal{I}_{\mathcal{K}}^{(i)} \cup \mathcal{I}_{\mathcal{UK}}^{(i)} \cup \mathcal{I}_{\mathcal{UC}}^{(i)}$ as follows:

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{(i)} &\triangleq \{j : \lambda_{ij} \text{ is known}\}, \\ \mathcal{I}_{\mathcal{UK}}^{(i)} &\triangleq \{j : \hat{\lambda}_{ij} \text{ is unknown}\}, \\ \mathcal{I}_{\mathcal{UC}}^{(i)} &\triangleq \{j : \tilde{\lambda}_{ij} \text{ is uncertain}\}, \end{aligned} \tag{4}$$

where each uncertain element is relabeled with a tilde "~". Moreover, if $\mathcal{I}_{\mathcal{K}}^{(i)} \neq \emptyset$ and $\mathcal{I}_{\mathcal{UC}}^{(i)} \neq \emptyset$, it is further described as

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{(i)} &= (\mathcal{K}_1, ..., \mathcal{K}_{m_i}),\ \forall 1 \leq m_i \leq N-2 \\ \mathcal{I}_{\mathcal{UC}}^{(i)} &= (\mathcal{U}_1, ..., \mathcal{U}_{n_i}),\ \forall 1 \leq n_i \leq N \end{aligned} \tag{5}$$

where $\mathcal{K}_s \in \mathbb{N}^+$, $s \in \{1, 2, \ldots, m_i\}$, and $\mathcal{U}_p \in \mathbb{N}^+, p \in \{1, 2, \ldots, n_i\}$ represents, respectively, the $s^{th}$ known element and $p^{th}$ uncertain element with the index $\mathcal{K}_{m_i}$ and $\mathcal{U}_{n_i}$ in the $i^{th}$ row of TRM $\mathbf{\Lambda}$. Also, we denote $\lambda_{\mathcal{K}}^{(i)} \triangleq \sum_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \lambda_{ij}$, $\lambda_{\mathcal{UC}}^{(i)} \triangleq \sum_{l \in \mathcal{I}_{\mathcal{UC}}^{(i)}} \tilde{\lambda}_{il}^r$, $\forall r = 1, ...M$ throughout the paper. Besides, if $\lambda_{ii}$ is unknown, it is essential to supply a lower bound $\lambda_d^{(i)}$ satisfying $\lambda_d^{(i)} \leq -(\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})$.

*Remark 3.* It is straightforward that the case $m_i = N-1$ should be excluded from (5), which means that if we know $N-1$ elements, the remaining unknown elements can be calculated via the property of TRM.

To describe the main objective more precisely, we now introduce the following definition for the underlying system. The more details can be referred to Boukas [2005] and the references therein.

*Definition 1.* (Boukas [2005]) System (1) is said to be stochastically stable if for $u(t) \equiv 0$ and every initial condition $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathcal{I}$, the following holds,

$$E\left\{\int_0^\infty \|x(t)\|^2 | x_0, r_0\right\} < \infty$$

The purposes of this paper are to derive the stochastic stability criteria for the system (1) when the statistics of the transition rates is defective as stated in (3) and to design a state-feedback stabilizing controller such that the resulting closed-loop system is stochastically stable. The mode-dependent controller is considered here with the form:

$$u(t) = K(r_t)x(t) \tag{6}$$

where $K_i$ ($\forall r_t = i \in \mathcal{I}$) is the controller gain to be determined.

The following Lemma on the stochastic stability of systems (1) is recalled for the developments in the later Section.

*Lemma 1.* (Boukas [2005]) System (1) is stochastically stable if and only if there exists a set of symmetric and positive-definite matrices $P_i, i \in \mathcal{I}$ satisfying

$$A_i^T P_i + P_i A_i + \mathcal{P}^{(i)} < 0 \tag{7}$$

where $\mathcal{P}^{(i)} \triangleq \sum_{j \in \mathcal{I}} \lambda_{ij} P_j$

## 3. STABILITY AND STABILIZATION

In this section, we will derive the stability and stabilization results based on Lemma 1 for the underlying systems. We first give the stability criteria for the unforced system (1) with $u(t) \equiv 0$.

The following theorem presents a necessary and sufficient condition on the stochastic stability of the considered systems with defective TRMs (3).

*Theorem 1.* Consider the unforced system (1) with the defective TRM (3). The corresponding system is stochastically stable if and only if there exists a set of matrices $P_i > 0$, $i \in \mathcal{I}$ such that $\forall i \in \mathcal{I}$,

$$\Omega_i - \lambda_{\mathcal{K}}^{(i)} P_j - \lambda_{\mathcal{UC}}^{(i)} P_j < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)},\ if\ i \in \mathcal{I}_{\mathcal{K}}^{(i)} \cup \mathcal{I}_{\mathcal{UC}}^{(i)} \tag{8}$$

$$\begin{aligned} \Omega_i + \lambda_d^{(i)} P_i - \lambda_d^{(i)} P_j - \lambda_{\mathcal{K}}^{(i)} P_j - \lambda_{\mathcal{UC}}^{(i)} P_j < 0, \\ \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, if\ i \in \mathcal{I}_{\mathcal{UK}}^{(i)} \end{aligned} \tag{9}$$

where $\Omega_i \triangleq A_i^T P_i + P_i A_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \mathcal{P}_{\mathcal{UC}}^{(i)}$ with

$$\mathcal{P}_{\mathcal{K}}^{(i)} \triangleq \sum_{j \in \mathcal{I}_{\mathcal{K}}^{(i)}} \lambda_{ij} P_j, \mathcal{P}_{\mathcal{UC}}^{(i)} \triangleq \sum_{l \in \mathcal{I}_{\mathcal{UC}}^{(i)}} \tilde{\lambda}_{il}^r P_l$$

and $\lambda_d^{(i)}$ represents a given lower bound of the unknown diagonal element if $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$.

**Proof.** We shall separate the proof into two cases, $i \in \mathcal{I}_{\mathcal{K}}^{(i)} \cup \mathcal{I}_{\mathcal{UC}}^{(i)}$ and $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, and bear in mind that system (1) is stochastically stable if and only if (7) holds.

1) Case I: $i \in \mathcal{I}_{\mathcal{K}}^{(i)} \cup \mathcal{I}_{\mathcal{UC}}^{(i)}$.

Note that $\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)} \leq 0$ in this case, then we only need to consider $\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)} < 0$ here, since $\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)} = 0$ means the $i^{th}$ row of the vertices $\mathbf{\Lambda}_r, r = 1, 2, \cdots, M$, is completely known.

Now we can rewrite the left-hand side of (7) as

$$\begin{aligned} \Theta_i &\triangleq A_i^T P_i + P_i A_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \hat{\lambda}_{ij} P_j \\ &\quad + \sum_{l \in \mathcal{I}_{\mathcal{UC}}^{(i)}} \left(\sum_{r=1}^M \alpha_r \tilde{\lambda}_{il}^r\right) P_l \\ &= \sum_{r=1}^M \alpha_r \left\{A_i^T P_i + P_i A_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \sum_{l \in \mathcal{I}_{\mathcal{UC}}^{(i)}} \tilde{\lambda}_{il}^r P_l \right. \\ &\quad \left. + \sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \hat{\lambda}_{ij} P_j \right\} \end{aligned}$$

where the elements $\hat{\lambda}_{ij}$, $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$ are unknown and $\sum_{r=1}^M \alpha_r \tilde{\lambda}_{il}^r$, $\forall l \in \mathcal{I}_{\mathcal{UC}}^{(i)}$ represents the uncertain element in the polytopic uncertainty description.

As $\sum_{r=1}^M \alpha_r = 1$ and $\alpha_r$ can take value arbitrarily in $[0, 1]$, we know that $\Theta_i < 0$ holds if and only if

$$A_i^T P_i + P_i A_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \mathcal{P}_{\mathcal{UC}}^{(i)} + \sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \hat{\lambda}_{ij} P_j < 0 \tag{10}$$

which can be further written as

$$\Omega_i - (\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)}) \sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\lambda}_{ij}}{-(\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})} P_j < 0$$

By $0 \leq \frac{\hat{\lambda}_{ij}}{-(\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})} \leq 1$ and $\sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\lambda}_{ij}}{-(\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})} = 1$, we know that the above inequality equals that

$$\sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}} \frac{\hat{\lambda}_{ij}}{-(\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})} \left(\Omega_i - (\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)}) P_j\right) < 0 \tag{11}$$

Thus, for $0 \le \hat{\lambda}_{ij} \le -(\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})$, (11) is equivalent to $\Omega_i - (\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})P_j < 0$, $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, which implies that, in the presence of both unknown elements $\hat{\lambda}_{ij}$ and uncertain elements $\tilde{\lambda}_{ij}$, the system is stable if and only if (8) holds.

2) Case II: $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$.

In this case, $\hat{\lambda}_{ii}$ is unknown, $\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)} \ge 0$ and $\hat{\lambda}_{ii} \le -\lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}$. Also, we only consider $\hat{\lambda}_{ii} < -\lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}$ here, since $\hat{\lambda}_{ii} = -\lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}$ means the $i^{th}$ row of the vertices $\mathbf{\Lambda}_r, r = 1, 2, \cdots, M$ is completely known.

Now the left-hand side of the stability condition in (7) can be rewritten as

$$\begin{aligned}\Theta_i &\triangleq A_i^T P_i + P_i A_i + \mathcal{P}_{\mathcal{K}}^{(i)} + \sum\nolimits_{l \in \mathcal{I}_{\mathcal{UC}}^{(i)}} \left( \sum_{r=1}^{M} \alpha_r \tilde{\lambda}_{il}^r \right) P_l \\ &\quad + \hat{\lambda}_{ii} P_i + \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \ne i} \hat{\lambda}_{ij} P_j \\ &= \sum_{r=1}^{M} \alpha_r \left( \Omega_i + \hat{\lambda}_{ii} P_i + \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \ne i} \hat{\lambda}_{ij} P_j \right)\end{aligned}$$

Likewise, as $\sum_{r=1}^{M} \alpha_r = 1$ and $\alpha_r$ can take value arbitrarily in $[0, 1]$, one has that $\Theta_i < 0$ holds if and only if

$$\Omega_i + \hat{\lambda}_{ii} P_i + \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \ne i} \hat{\lambda}_{ij} P_j < 0$$

which can be rewritten as

$$\begin{aligned}&\Omega_i + \hat{\lambda}_{ii} P_i + (-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}) \\ &\times \sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \ne i} \frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}} P_j < 0\end{aligned}$$

Considering the fact $0 \le \frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}} \le 1$ and $\sum_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \ne i} \frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}} = 1$, we know that the above inequality equals that

$$\begin{aligned}&\sum\nolimits_{j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, j \ne i} \frac{\hat{\lambda}_{ij}}{-\hat{\lambda}_{ii} - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}} (\Omega_i + \hat{\lambda}_{ii} P_i \\ &- \hat{\lambda}_{ii} P_j - \lambda_{\mathcal{K}}^{(i)} P_j - \lambda_{\mathcal{UC}}^{(i)} P_j) < 0 \qquad (12)\end{aligned}$$

which means that (12) is equivalent to $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, $j \ne i$,

$$\Omega_i + \hat{\lambda}_{ii} P_i - \hat{\lambda}_{ii} P_j - \lambda_{\mathcal{K}}^{(i)} P_j - \lambda_{\mathcal{UC}}^{(i)} P_j < 0 \qquad (13)$$

As $\hat{\lambda}_{ii}$ is lower bounded by $\lambda_d^{(i)}$, we have

$$\lambda_d^{(i)} \le \hat{\lambda}_{ii} < -\lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}$$

which implies that $\hat{\lambda}_{ii}$ may take any value between $[\lambda_d^{(i)}, \delta - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}]$ for some $\delta < 0$ arbitrarily small. Then $\hat{\lambda}_{ii}$ can be further written as a convex combination

$$\hat{\lambda}_{ii} = \beta\delta - \beta\lambda_{\mathcal{K}}^{(i)} - \beta\lambda_{\mathcal{UC}}^{(i)} + (1 - \beta)\lambda_d^{(i)}$$

where $\beta$ takes value arbitrarily in $[0, 1]$. Thus, (13) holds if and only if $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, $j \ne i$,

$$\begin{aligned}&\Omega_i + \left( \delta - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)} \right) (P_i - P_j) \\ &- \lambda_{\mathcal{K}}^{(i)} P_j - \lambda_{\mathcal{UC}}^{(i)} P_j < 0 \qquad (14)\end{aligned}$$

and

$$\Omega_i + \lambda_d^{(i)} P_i - \lambda_d^{(i)} P_j - \lambda_{\mathcal{K}}^{(i)} P_j - \lambda_{\mathcal{UC}}^{(i)} P_j < 0 \qquad (15)$$

simultaneously hold. Since $\delta$ is arbitrarily small, (14) holds if and only if

$$\begin{aligned}&\Omega_i + \left( -\lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)} \right) (P_i - P_j) \\ &- \lambda_{\mathcal{K}}^{(i)} P_j - \lambda_{\mathcal{UC}}^{(i)} P_j < 0\end{aligned}$$

i.e.,

$$\Omega_i - \lambda_{\mathcal{K}}^{(i)} P_i - \lambda_{\mathcal{UC}}^{(i)} P_i < 0$$

which is the case in (15) when $j = i$, $\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}$. Hence (13) is equivalent to (9).

Therefore, in the presence of both unknown elements and uncertain elements, one can readily conclude that the system is stable if and only if (8) and (9) hold for $i \in \mathcal{I}_{\mathcal{K}}^{(i)} \cup \mathcal{I}_{\mathcal{UC}}^{(i)}$ and $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, respectively.

Now let us consider the stabilization problem of system (1) with control input $u(t)$. The following theorem provides necessary and sufficient conditions for the existence of a mode-dependent stabilizing controller with the form (6).

*Theorem 2.* Consider system (1) with the defective TRM (3). If there exist matrices $X_i > 0$ and $Y_i$, $\forall i \in \mathcal{I}$, such that

$$\begin{bmatrix} \Phi_i + \Xi_i & \mathcal{T}^{(i)} & \digamma_i X_i \\ * & -\mathcal{X}^{(i)} & 0 \\ * & * & -X_j \end{bmatrix} < 0, \forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)} \qquad (16)$$

where

$$\Xi_i \triangleq \begin{cases} \lambda_{ii} X_i, \text{ if } i \in \mathcal{I}_{\mathcal{K}}^{(i)} \\ \tilde{\lambda}_{ii}^r X_i, \text{ if } i \in \mathcal{I}_{\mathcal{UC}}^{(i)} \\ \lambda_d^{(i)} X_i, \text{ if } i \in \mathcal{I}_{\mathcal{UK}}^{(i)} \end{cases}$$

$$\Phi_i \triangleq A_i X_i + X_i A_i^T + B_i Y_i + Y_i^T B_i^T$$

$$\digamma_i \triangleq \begin{cases} \sqrt{-\lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}}, \text{ if } i \in \mathcal{I}_{\mathcal{K}}^{(i)} \cup \mathcal{I}_{\mathcal{UC}}^{(i)} \\ \sqrt{-\lambda_d^{(i)} - \lambda_{\mathcal{K}}^{(i)} - \lambda_{\mathcal{UC}}^{(i)}}, \text{ if } i \in \mathcal{I}_{\mathcal{UC}}^{(i)} \end{cases}$$

and

$$\begin{aligned}&\mathcal{X}^{(i)} \triangleq diag \left[ X_{\mathcal{K}_1}, \ldots, X_{\mathcal{K}_{m_i}}, X_{\mathcal{U}_1}, \ldots, X_{\mathcal{U}_{n_i}} \right], \\ &\mathcal{T}^{(i)} \triangleq \Big[ \sqrt{\lambda_{i\mathcal{K}_1}} X_i, \ldots, \sqrt{\lambda_{i\mathcal{K}_{m_i}}} X_i, \sqrt{\tilde{\lambda}_{i\mathcal{U}_1}^r} X_i, \\ &\qquad \ldots, \sqrt{\tilde{\lambda}_{i\mathcal{U}_{n_i}}^r} X_i \Big] \qquad (17)\end{aligned}$$

and $\forall s \in \{1, 2, \ldots, m_i\}$, $\mathcal{K}_s \ne i$, $\forall p \in \{1, 2, \ldots, n_i\}$, $\mathcal{U}_p \ne i$, with $\mathcal{K}_s$ and $\mathcal{U}_p$ described in (5). Then there exists a mode-dependent stabilizing controller of the form in (6) such that the closed-loop system is stochastically stable. In addition, if the conditions of (16) have a solution, an admissible controller gain is proposed by

$$K_i = Y_i X_i^{-1} \qquad (18)$$

**Proof.** Consider system (1) with the control input (6) and replace $A_i$ by $A_i + B_i K_i$ in (8)–(9), respectively. Then, if $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$, performing a congruence transformation to (8) by $P_i^{-1}$, we can obtain

$$(A_i + B_iK_i)\, P_i^{-1} - P_i^{-1}(\lambda_{\mathcal{K}}^{(i)} + \lambda_{\mathcal{UC}}^{(i)})P_jP_i^{-1} + P_i^{-1}(\mathcal{P}_{\mathcal{K}}^{(i)} + \mathcal{P}_{\mathcal{UC}}^{(i)})P_i^{-1} + P_i^{-1}(A_i + B_iK_i)^T < 0 \quad (19)$$

Setting $X_i \triangleq P_i^{-1}$, $Y_i \triangleq K_iX_i$ and considering (17), by Schur complement, one can obtain that (19) is equivalent to the case $i \in \mathcal{I}_{\mathcal{K}}^{(i)}$ in (16). In a similar way, if $i \in \mathcal{I}_{\mathcal{UC}}^{(i)}$ or $i \in \mathcal{I}_{\mathcal{UK}}^{(i)}$, the corresponding cases in (16) can be worked out from (8) or (9), respectively. Meanwhile, due to $Y_i = K_iX_i$, the desired controller gain is given by (18).

*Remark 4.* The defective TRMs (3) considered in Theorem 2 contain both uncertain TPs and unknown TPs. If we consider one of them, i.e., only uncertain TPs or partially unknown TPs exist in (3), then we can obtain two corollaries from Theorem 2, respectively, in what follows.

*Corollary 1.* Consider system (1) with partially unknown TPs, i.e., $\mathcal{I}_{\mathcal{UC}}^{(i)} = \emptyset$ in the defective TRM (2). If there exist matrices $X_i > 0$ and $Y_i$, $\forall i \in \mathcal{I}$, such that

$$\begin{bmatrix} \Phi_i + \lambda_{ii}X_i & \mathcal{T}_{\mathcal{K}}^{(i)} & \sqrt{-\lambda_{\mathcal{K}}^{(i)}}X_i \\ * & -\mathcal{X}_{\mathcal{K}}^{(i)} & 0 \\ * & * & -X_j \end{bmatrix} < 0,$$
$$\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, if\ i \in \mathcal{I}_{\mathcal{K}}^{(i)} \quad (20)$$

$$\begin{bmatrix} \Phi_i + \lambda_d^{(i)}X_i & \mathcal{T}_{\mathcal{K}}^{(i)} & \sqrt{-\lambda_d^{(i)} - \lambda_{\mathcal{K}}^{(i)}}X_i \\ * & -\mathcal{X}_{\mathcal{K}}^{(i)} & 0 \\ * & * & -X_j \end{bmatrix} < 0,$$
$$\forall j \in \mathcal{I}_{\mathcal{UK}}^{(i)},\ if\ i \in \mathcal{I}_{\mathcal{UK}}^{(i)} \quad (21)$$

where $\Phi_i \triangleq A_iX_i + X_iA_i^T + B_iY_i + Y_i^TB_i^T$ and

$$\mathcal{X}_{\mathcal{K}}^{(i)} \triangleq diag\left[X_{\mathcal{K}_1}, \ldots, X_{\mathcal{K}_{m_i}}\right],$$
$$\mathcal{T}_{\mathcal{K}}^{(i)} \triangleq \left[\sqrt{\lambda_{i\mathcal{K}_1}}X_i, \ldots, \sqrt{\lambda_{i\mathcal{K}_{m_i}}}X_i\right]$$

and $\forall s \in \{1, 2, \ldots, m_i\}$, $\mathcal{K}_s \neq i$, with $\mathcal{K}_s$ described in (5). Then there exists a mode-dependent stabilizing controller of the form in (6) such that the closed-loop system is stochastically stable. In addition, if (20)–(21) have a solution, an admissible controller gain is proposed by (18).

*Corollary 2.* Consider system (1) with uncertain TPs, i.e., $\mathcal{I}_{\mathcal{UK}}^{(i)} = \emptyset$ in the defective TRM (3). If there exist matrices $X_i > 0$ and $Y_i$, $\forall i \in \mathcal{I}$, such that

$$\begin{bmatrix} \Phi_i + \Xi_i & \mathcal{T}^{(i)} \\ * & -\mathcal{X}^{(i)} \end{bmatrix} < 0 \quad (22)$$

where

$$\Xi_i \triangleq \begin{cases} \lambda_{ii}X_i, & \text{if } i \in \mathcal{I}_{\mathcal{K}}^{(i)} \\ \tilde{\lambda}_{ii}^rX_i, & \text{if } i \in \mathcal{I}_{\mathcal{UC}}^{(i)} \end{cases}$$

and $\Phi_i, \mathcal{X}^{(i)}$ and $\mathcal{T}^{(i)}$ are denoted in Theorem 2, and $\forall s \in \{1, 2, \ldots, m_i\}$, $\mathcal{K}_s \neq i$, $\forall p \in \{1, 2, \ldots, n_i\}$, $\mathcal{U}_p \neq i$, with $\mathcal{K}_s$ and $\mathcal{U}_p$ described in (5). Then there exists a mode-dependent stabilizing controller of the form in (6) such that the closed-loop system is stochastically stable. In addition, if the conditions of (22) have a solution, an admissible controller gain is proposed by (18).

*Remark 5.* The proofs for Corollaries 1 and 2 can be obtained from Theorem 2 by removing the set of uncertain elements, or the set of unknown elements, respectively. It is straightforward that the case of defective statistics and the corresponding MJLSs addressed in Theorem 2 are more general.

*Remark 6.* Note that in Theorem 2, the level of the defective statistics varies as the unknown elements become uncertain or the intervals of the uncertain elements become tighter. Then it is natural to conjecture that there will exist a monotonicity with respect to the system performance (or the existence of the stabilizing controllers in this paper), as the level of the defective TRMs varies, which we will verify via numerical examples in next section.

## 4. NUMERICAL EXAMPLES

In this section, two numerical examples are provided to show the validity of the theoretical results. In particular, we will also verify the monotonicity that we conjectured when the levels of the defective statistics varies. For the conciseness, we denote the $s^{th}$ row of the $r^{th}$ vertices in the polytope uncertainty description as $\mathbf{\Lambda}_r^s$, $\forall s = 1, .., N, \forall r = 1, ..., M$.

*Example 1.* Consider MJLS (1) with four operation modes and the following system matrices:

$$A_1 = \begin{bmatrix} -27 & -13.5 \\ 18 & 12.6 \end{bmatrix},\ A_2 = \begin{bmatrix} 28.8 & -5.9 \\ 18 & 25.2 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -3.6 & 1.8 \\ 18 & 18 \end{bmatrix},\ A_4 = \begin{bmatrix} 18 & -4.1 \\ 18 & -19.8 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$B_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ B_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The TRM (3) consists of two vertices $\mathbf{\Lambda}_i, i = 1, 2$, where $\mathbf{\Lambda}_1^1$ and $\mathbf{\Lambda}_2^1$ are given by

$$\mathbf{\Lambda}_1^1 = [-3.3, \hat{\lambda}_{12}, 1.6, \hat{\lambda}_{14}], \mathbf{\Lambda}_2^1 = [-3.3, \hat{\lambda}_{12}, 3.2, \hat{\lambda}_{14}] \quad (23)$$

and other rows $\mathbf{\Lambda}_1^s$, $\mathbf{\Lambda}_2^s$, $s = 2, 3, 4$ are same with elements $\forall r = 1, 2$

$$\mathbf{\Lambda}_r^2 = [\hat{\lambda}_{21}, \hat{\lambda}_{22}, \hat{\lambda}_{23}, 0.5],$$
$$\mathbf{\Lambda}_r^3 = [\hat{\lambda}_{31}, 0.6, -2.5, \hat{\lambda}_{34}],$$
$$\mathbf{\Lambda}_r^4 = [\hat{\lambda}_{41}, \hat{\lambda}_{42}, \hat{\lambda}_{43}, -1.2] \quad (24)$$

For convenient verifications, we rewrite the true TRM $\mathbf{\Lambda}$ from (23) and (24) as the following description:

$\mathbf{\Lambda} =$

| $Mode$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | $-3.3$ | $\hat{\lambda}_{12}$ | $[1.6, 3.2]$ | $\hat{\lambda}_{14}$ |
| 2 | $\hat{\lambda}_{21}$ | $\hat{\lambda}_{22}$ | $\hat{\lambda}_{23}$ | 0.5 |
| 3 | $\hat{\lambda}_{31}$ | 0.6 | $-2.5$ | $\hat{\lambda}_{34}$ |
| 4 | $\hat{\lambda}_{41}$ | $\hat{\lambda}_{42}$ | $\hat{\lambda}_{43}$ | $-1.2$ |

where the uncertain element $\tilde{\lambda}_{13}$ has a range $[1.6, 3.2]$. The purpose here is to verify the monotonicity with respect to the lower bound $\lambda_d^{(2)}$ of the unknown element $\hat{\lambda}_{22}$ when the range of the uncertain elements $\tilde{\lambda}_{13}$ become tighter. Such a change of the range can be realized by assigning different values for $\tilde{\lambda}_{13}$ in $\Lambda_r^1, r = 1, 2$. Firstly, it can be checked that the open-loop system is unstable based on Theorem 1 for any $\lambda_d^{(2)} \in (-\infty, -0.5]$. Then by Theorem 2, together with using the bisection method, we can obtain the minimal value of $\lambda_d^{(2)}$, below which the stabilizing controller will not exist. Table 2 gives the different minimal values of $\lambda_d^{(2)}$ corresponding to different ranges of $\tilde{\lambda}_{13}$.

| *system* | $\hat{\lambda}_{13}$ | $\lambda_{d\min}^{(2)}$ |
|---|---|---|
| 1 | $[1.6, 3.2]$ | $-6.2682$ |
| 2 | $[1.9, 2.9]$ | $-7.6323$ |
| 3 | $[2.2, 2.6]$ | $-9.8732$ |

Table 2. Minimal values of $\lambda_d^{(2)}$

From Table 2, a monotonicity can be observed that when the range of $\tilde{\lambda}_{13}$ becomes tighter, the value of $\lambda_{d\min}^{(2)}$ becomes smaller, which means that the admissible stabilizing controller exists for the system within a larger domain of unknown element $\hat{\lambda}_{22}$. The controller gains are omitted here due to space limit.

*Example 2.* Consider MJLS (1) with four operation modes and the following system matrices:

$$A_1 = \begin{bmatrix} -22.5 & -12.6 \\ 18.7 & 11.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 27.2 & -5.8 \\ 18 & 25.2 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -3.4 & 1.7 \\ 15.3 & 17 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 17 & -3.9 \\ 17 & -18.7 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1.1 \\ 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$B_3 = \begin{bmatrix} 1 \\ -2.1 \end{bmatrix}, \ B_4 = \begin{bmatrix} -0.9 \\ 1 \end{bmatrix}.$$

and the TRM is considered as

$$\mathbf{\Lambda} = \begin{array}{|c||c|c|c|c|} \hline Mode & 1 & 2 & 3 & 4 \\ \hline 1 & -1.3 & \hat{\lambda}_{12} & \hat{\lambda}_{12} & \hat{\lambda}_{12} \\ \hline 2 & \hat{\lambda}_{21} & \hat{\lambda}_{22} & \hat{\lambda}_{23} & 0.5 \\ \hline 3 & \hat{\lambda}_{31} & \hat{\lambda}_{32} & -4.5 & \hat{\lambda}_{34} \\ \hline 4 & \hat{\lambda}_{41} & \hat{\lambda}_{42} & \hat{\lambda}_{43} & -1.2 \\ \hline \end{array} \tag{25}$$

The purpose of this example is to verify the monotonicity when unknown elements become uncertain ones. We assign the unknown elements $\hat{\lambda}_{31}$ and $\hat{\lambda}_{32}$ with different ranges, which can be realized by further giving different vertices based on $\mathbf{\Lambda}$ in (25). By Theorem 2 and bisection method, we can obtain the minimal values of $\lambda_d^{(2)}$ below which the stabilizing controller does not exist for the system. The computation results are listed in Table 2. It can be seen that when more unknown elements become uncertain, the minimal value of $\lambda_d^{(2)}$ decreases, which means the solutions of stabilizing controllers are feasible within a larger domain of $\hat{\lambda}_{22}$.

| *system* | $\hat{\lambda}_{31}$ | $\hat{\lambda}_{32}$ | $\lambda_{d\min}^{(2)}$ |
|---|---|---|---|
| 1 | ? | ? | $-3.7930$ |
| 2 | $[1.6, 2.0]$ | ? | $-5.4071$ |
| 3 | $[1.6, 2.0]$ | $[1.8, 2.2]$ | $-8.5680$ |

Table 3. Unknown elements turning into uncertain ones

In Examples 1 and 2, the validity of the obtained Theorem 2 is demonstrated, that is, the stabilizing controllers for the underlying systems exist despite the defective statistics of modes transitions, which contains both uncertain elements and unknown elements in the corresponding TRM. In addition, the level of defective statistics about modes transitions decreases, as the the ranges of uncertain elements are tighter, or as the unknown elements become uncertain, i.e., with further contrived ranges. It can be concluded from Tables 2 and 3 that more knowledge on the statistics of modes transitions are available to the designers, the relevant system performance (the existence of admissible stabilizing controllers here) will be improved.

## 5. CONCLUSIONS

In this paper, the problem of the stabilization of MJLSs with a class of defective description of mode transitions was investigated. The defective statistics about modes transitions in the study composite the recent separate studies, the uncertain TPs and partially unknown TPs and the underlying systems are more general accordingly. By using the property of TRM and the convexity of uncertain domains, the necessary and sufficient conditions for the stability and stabilization of the underlying system are obtained. A monotonicity, in concern of the existence of the admissible stabilizing controller, is observed when the unknown elements become uncertain or the ranges of the uncertain ones become tighter. It is expected that the idea and approaches behind the paper can be used to other analysis and synthesis problems of the underlying systems.

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