A BRL for A Class of Discrete-time Markov Jump Linear System with Piecewise-Constant TPs ⋆

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Abstract: This paper aims to obtain a Bounded Real Lemma (BRL) for a class of Markov jump linear systems (MJLSs) with time-varying transition probabilities (TPs) in discrete-time domain. The time-varying character of TPs is considered as piecewise-constant and the variation of TP matrices is subject to average dwell time (ADT) switching, i.e., the number of switches in a finite interval is bounded and the *average* time between two consecutive switchings of TP matrices is not less than a constant. Combining the Lyapunov function approach and the linear matrix inequality technique, a BRL for the underlying system is derived in order to check whether the corresponding system is stochastically stable and has a guaranteed H_∞ noise-attenuation performance index scheduled based on the variation of TP matrices. A numerical example is provided to demonstrate that the method obtained in this paper is a significant improvement over previous one.

Keywords: Average dwell time (ADT), H_∞ performance analysis, Linear matrix inequality, Markov jump linear systems (MJLSs), Piecewise-Constant TPs.

1. INTRODUCTION

Markov jump systems (MJSs) rapidly developed and flourished due to the powerful modeling ability of Markov process in many fields, such as aerospace industry, de Farias et al. [2005], communication systems, Zhang et al. [2005], Seiler and Sengupta [2005], biology and medicine, Chan et al. [2002], Ullah and Wolkenhauer [2007], economics, Fingleton [1997], Gonzalez et al. [2005], etc. The system is hybrid in essence, where the continuous and discrete dynamics are, respectively, described by a set of classical differential (or difference) equations and an attached Markov stochastic process (or Markov chain) governing the transitions among them. As a crucial factor, the transition rates (TRs) or transition probabilities (TPs) in the Markov process or Markov chain determine system behavior and performance. Over the past decades, the system has been mainly studied upon the assumption that the TRs (or TPs) are certain and completely known. Recent investigations considering MJSs with uncertain or partially unknown TRs (or TPs) have also been reported in literature, see for example, Zhang et al. [2008], Xiong

et al. [2005], Zhang and Boukas [2009]. Yet, so far, almost all the available analysis and synthesis results assume that the Markov processes or Markov chains in the underlying systems are time-invariant.

However, the assumption is not true in practice. A typical example can be found in Internet-based Networked control systems (NCS). It is well-known that the packet dropouts and channel delays in Internet can be modeled by Markov chains and the resulting systems are accordingly the traditional MJSs, Seiler and Sengupta [2005], Zhang et al. [2005], Krtolica et al. [1976]. But for the Internet nowadays, the delays or packet losses are distinct at different periods, the resulting TP matrix may vary throughout the running time of the modeled system. For another example, we refer to the VTOL (vertical take-off landing) helicopter system, Narendra and Tripathi [1973], where the airspeed variations involved in the system matrices are ideally modeled as time-invariant Markov chain, de Farias et al. [2000]. But all the probabilities of the jumps among multiple airspeeds will not be fixed when external environment (like weather) changes. The similar phenomenon also arises in other practical systems, Krtolica et al. [1976].

The existence of variations in TRs (or TPs) challenges the traditional control approaches for MJSs, and the integrity in control theory suggests the desirability of allowing the TRs (or TPs) to vary. By time-varying character, we mean that the TR or TP matrix in the Markov process or Markov chain changes depending on time. In fact, there are some burgeoning studies on the stability of a class of MJLSs with uncertain TRs (TPs), where the uncertainties

can be thought as the ones from either modeling or the variations in practice, e.g., Xiong et al. [2005], Karan et al. [2006], Boukas et al. [1999] . Accordingly, the analogous knowledge for uncertain dynamic systems is absorbed and the robust methodologies are adopted to solve the norm-bounded or polytopic uncertainties likely existing in the TRs. It is also observed in Xiong et al. [2005] that uncertainties in TRs may lead to instability, which directly means if TRs vary in the system running period, the desired stability or performance might be lost. The method avoids the measurement of the variations in TRs, but probably conservative. Therefore, it is incomplete to only study the uncertain TPs. Recently, a class of time-varying Markov processes (or chains) subject to deterministic switching signal have been proposed, Bolzern et al. [2010]. It implies that the TRs (or TPs) therein are varying but invariant within an interval. The so-called MJLSs with piecewise-constant TPs is supposed to contain a finite set of consecutive time-invariant Markov processes (or chains) with different intervals, longer or shorter. Hence, the whole system can be viewed as a switched system, where each subsystem is the usual MJS. An illustration about a possible variation of TP matrices subject to deterministic switching signal and the corresponding possible path of modes evolution is given in Fig. 1.

On the other hand, in the area of determined switched systems, rapid progress has shown that another class of switching signals, the so-called average dwell time(ADT) switching is more general and flexible than the DT switching, Liberzon [2003], in related stability analyses and control syntheses. The reason is that, in the ADT switching, the number of switches in a finite interval is bounded and the *average* time between consecutive switching is not less than a constant, Hespanha and Morse [1999], Liberzon [2003]. However, in Bolzern et al. [2010], the switching signal is still restricted to the DT switching, the advanced ADT switching rule has not been included to investigate the control problems of the MJLSs with piecewise-constant TPs.

In this paper, we investigate the H_∞ performance analysis problem for a class of discrete-time MJLSs with piecewise-constant TPs. The variation of piecewise-constant TP matrices is determined by a high-level ADT switching signal. Utilizing the Lyapunov function, a BRL in discrete-time of the underlying system is derived. The remainder of the paper is organized as follows. In section 2, the basic definitions on stability and H_∞ performance of the underlying system are represented, and the H_∞ performance analysis problem is formulated. The BRL is derived in Section 3. A numerical example for illustration is given in Section 4 and we conclude the paper in Section 5.

Remark 1. It is noticed that in this paper the high-level switching signal determining the variation of TPs is a typical slow switching signal, the ADT switching. However, in Zhang [2009], the variations of TPs are subject to two types of switching signal: the fast switching and the stochastic switching. Therefore, the variation of TPs considered in this paper is essentially different from those in Zhang [2009].

Notation: The notation used in this paper is standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$

denotes the n dimensional Euclidean space, the notation $\|\cdot\|$ refers to the Euclidean vector norm; $\mathbb{N}^+$ represents the sets of positive integers. $l_2[0,\infty)$ is the space of summable infinite sequence over $[0,\infty)$ and for $w(k) \in l_2[0,\infty)$, its norm is given by $\|w\|_2 = \sqrt{\sum_{k=0}^{\infty} |w(k)|^2}$. For notation $(\Omega, \mathcal{F}, \mathcal{P})$, Ω represents the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. $E[\cdot]$ stands for the mathematical expectation and for $z(k) \in l_2((\Omega, \mathcal{F}, \mathcal{P}), [0,\infty))$, its norm is given by $\|z\|_{E_2} = \sqrt{E\left[\sum_{k=0}^{\infty} |z(k)|^2\right]}$. In addition, in symmetric block matrices or long matrix expressions, we use * as an ellipsis for the terms that are introduced by symmetry and $diag\{\cdots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The notation $P > 0 (\geq 0)$ means P is real symmetric positive (semi-positive) definite, M_i is adopted to denote $M(i)$ for brevity. I and 0 represent respectively, identity matrix and zero matrix.

2. PRELIMINARIES AND PROBLEM FORMULATION

Fix the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and consider the following discrete-time MJLS:

$$\begin{aligned} x(k+1) &= A(r_k)x(k) + E(r_k)w(k) \\ z(k) &= C(r_k)x(k) + F(r_k)w(k) \end{aligned} \quad (1)$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ is the disturbance input which belongs to $l_2[0,\infty)$ and $z(k) \in \mathbb{R}^{n_z}$ is the output vector. The stochastic process $\{r_k, k \geq 0\}$, taking values in a finite set $\mathcal{I} \triangleq \{1, \ldots, N\}$, governs the switching among the different system modes with the following mode transition probabilities:

$$\Pr(r_k = j | r_k = i) = \pi_{ij}^{(\sigma_k)}$$

where $\pi_{ij}^{(\sigma_k)} \geq 0$, $\forall i, j \in \mathcal{I}$ denotes the transition probability (TP) from mode i to mode j at time $k \in [k_l, k_{l+1})$, and $\sum_{j=1}^{N} \pi_{ij}^{(\sigma_k)} = 1$ for all $i \in \mathcal{I}$. Here, by k, we mean that the TPs are time-varying, meanwhile, we assume that σ_k is a piecewise constant function of time k. Furthermore, the TP matrix $\mathbf{\Pi}^{(\sigma_k)}$ can be defined by:

$$\mathbf{\Pi}^{(\sigma_k)} = \begin{bmatrix} \pi_{11}^{(\sigma_k)} & \pi_{12}^{(\sigma_k)} & \cdots & \pi_{1N}^{(\sigma_k)} \\ \pi_{21}^{(\sigma_k)} & \pi_{22}^{(\sigma_k)} & \cdots & \pi_{2N}^{(\sigma_k)} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1}^{(\sigma_k)} & \pi_{N2}^{(\sigma_k)} & \cdots & \pi_{NN}^{(\sigma_k)} \end{bmatrix} \quad (2)$$

The set $\mathcal{I}$ contains N modes of system (1) and for $r_k = i, i \in \mathcal{I}$, the system matrices of the ith mode are denoted by $(A_i, B_i, C_i, D_i, E_i, F_i)$, which are real known with appropriate dimensions.

As a high-level signal to determine the time-varying property of r_k, we assume that σ_k vary in another finite set $\mathcal{M} \triangleq \{1, \ldots, M\}$, $M > 1$, without loss of generality. Specifically, σ_k is a given initial condition sequence, s denoting σ_k for simplicity. At an arbitrary time k, σ may be dependent on k or $x(k)$, or both, or other logic rules. For a switching sequence $k_0 < k_1 < k_2 < \ldots$, σ is continuous from right everywhere and may be either autonomous or

controlled. When $k \in [k_l, k_{l+1})$, we say that the $\sigma_{k_l}^{th}$ TP matrix is active.

To more precisely describe the main objective, we also introduce the following definitions for system (1).

Definition 1. (Hespanha and Morse [1999]) For switching signal σ and any $K > k > k_0$, let $N_\sigma(K, k)$ be the switching numbers of σ over the interval $[k, K)$. If for any given $N_0 > 0$ and $\tau_a > 0$, we have $N_\sigma(K, k) \le N_0 + (K - k)/\tau_a$, then τ_a and N_0 are called average dwell time and the chatter bound, respectively.

Definition 2. (Boukas [2005]) System (1) is said to be internally stochastically stable if for $u(k) \equiv 0, w(k) \equiv 0, k \ge 0$ and every initial condition $x_0 \in \mathbb{R}^{n_x}$ and $r_0 \in \mathcal{I}, \sigma_0 \in \mathcal{M}$, the following holds:

$$E\left\{ \sum_{k=0}^{\infty} \|x(k)\|^2 | x_0, r_0, \sigma_0 \right\} < \infty$$

Definition 3. (Boukas [2005]) Given a scalar $\gamma > 0$, system (1) is said to be stochastically stable and has an H_∞ noise attenuation performance index γ, if it is stochastically stable and under zero initial condition, $\|z\|_{E_2} < \gamma \|w\|_2$ hold for all nonzero $w_k \in l_2[0, \infty)$.

Thus, the problem to be addressed in the paper is: consider system (1) with piecewise-constant TP matrices (2), derive a BRL for the system with an admissible ADT condition to check whether the system is stochastically stable and achieve a guaranteed H_∞ noise-attenuation performance scheduled based on the variation of TP matrices.

Before proceeding further, we present the following result on the H_∞ noise-attenuation performance analysis for system (1) in Costa et al. [2005] for later use.

Lemma 1. System (1) with time-invariant TPs is internally stochastically stable and has a guaranteed H_∞ performance index γ if and only if there exists a set of symmetric and positive-definite matrices $P_i, \forall r_k = i, i \in \mathcal{I}$ satisfying:

$$\begin{bmatrix} -\bar{\mathcal{P}}_i & 0 & \tilde{\mathcal{P}}_i A_i & \tilde{\mathcal{P}}_i E_i \\ * & -I & C_i & F_i \\ * & * & -P_i & 0 \\ * & * & * & \gamma^2 I \end{bmatrix} < 0$$

where $\tilde{\mathcal{P}}_i \triangleq \sum_{j=1}^{N} \pi_{ij} P_j$.

3. BOUNDED REAL LEMMA

In this section, we will first develop a BRL for a class of MJLSs with uncertain TPs, and further give a BRL for the underlying system with piecewise-constant TPs subject to ADT switching.

In Lemma 1, we have assumed that the TP was free of uncertainties, which is not real in practice since it is always difficult to get the exact TP matrix. In the following proposition, we will try to take account of the uncertainties that may influence the TPs and establish equivalent result to Lemma 1. The uncertainties we will consider for the TP matrix are of polytopic ones.

As described in De Souza et al. [2006], it is assumed that TP matrix $\Pi = [\pi_{ij}]_{N \times N}$ belongs to a given polytope P_Π with vertices $\Pi_m, m = 1, 2, \ldots s$, i.e.,

$$P_\Pi \triangleq \left\{ \Pi | \Pi = \sum_{m=1}^{s} \alpha_m \Pi_m; 0 < \alpha_m < 1, \sum_{m=1}^{s} \alpha_m = 1 \right\}$$

where α_m means the uncertain parameter, s denoting the total number of the vertices, is a given positive integer, and Π_m is known TP matrix whose expression is given by:

$$\mathbf{\Pi}_m = \begin{bmatrix} \pi_{11}^m & \pi_{12}^m & \cdots & \pi_{1N}^m \\ \pi_{21}^m & \pi_{22}^m & \cdots & \pi_{2N}^m \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{N1}^m & \pi_{N2}^m & \cdots & \pi_{NN}^m \end{bmatrix}$$

Furthermore, adopting the robust methodologies, a BRL for a class of MJLSs with uncertain TPs can be derived, which is obtained in the following proposition.

Proposition 1. Consider system (1) with polytopic uncertain TPs. If there exist matrices $P_i > 0, \forall r_k = i, i \in \mathcal{I}$, such that the following holds for all admissible uncertainties:

$$\Psi_i^m \triangleq \begin{bmatrix} -\tilde{\mathcal{P}}_i^m & 0 & \tilde{\mathcal{P}}_i^m A_i & \tilde{\mathcal{P}}_i^m E_i \\ * & -I & C_i & F_i \\ * & * & P_i & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \qquad (3)$$

where $\tilde{\mathcal{P}}_i^m \triangleq \sum_{j=1}^{N} \pi_{ij}^m P_j$, then the system is stochastically stable and has a guaranteed H_∞ performance index γ.

Proof. Construct a Lyapunov function as
$$V(x_k, r_k) = x_k^T P(r_k) x_k$$

Then, from the point $(x_k = x, r_k = i), \forall r_k = i, i \in \mathcal{I}$, we know that for system (1):

$$\Delta V(x_k, i) = E[V(x_{k+1}, r_{k+1}) | x, i] - V(x_k, i)$$
$$= x_{k+1}^T \left(\sum_{j=1}^{N} \sum_{m=1}^{s} \alpha_m \pi_{ij}^m P_j \right) x_{k+1} - x_k^T P_i x_k$$
$$= \sum_{m=1}^{s} \alpha_m \{ x_k^T \left[A_i^T \tilde{\mathcal{P}}_i^m A_i - P_i \right] x_k$$
$$+ 2 x_k^T A_i^T \tilde{\mathcal{P}}_i^m E_i w_k + w_k^T E_i^T \tilde{\mathcal{P}}_i^m E_i w_k \}$$

where $\tilde{\mathcal{P}}_i^m \triangleq \sum_{j=1}^{N} \pi_{ij}^m P_j$, $\sum_{m=1}^{s} \alpha_m \pi_{ij}^m$ represents the uncertain element in the polytopic uncertainty description.

As $\sum_{m=1}^{s} \alpha_m = 1$ and α_m can take value arbitrarily in $[0, 1]$, when $w(k) \equiv 0$, if $A_i^T \tilde{\mathcal{P}}_i^m A_i - P_i < 0$, one has $\Delta V(x_k, i) < 0$. Following a similar vein in the proof of Theorem 1 in Boukas and Liu [2001], it can be shown that $E\left\{ \sum_{k=0}^{\infty} \|x_k\|^2 x_0, r_0 \right\} < \infty$, that is, the system is stochastically stable. By Schur complement, $A_i^T \tilde{\mathcal{P}}_i^m A_i - P_i < 0$ is equivalent to

$$\begin{bmatrix} -\tilde{\mathcal{P}}_i^m & \tilde{\mathcal{P}}_i^m A_i \\ * & -P_i \end{bmatrix} < 0 \qquad (4)$$

Now, to establish the H_∞ performance analysis criterion for system (1), consider the following performance index:

$$J \triangleq E \left\{ \sum_{k=0}^{\infty} \left[z_k^T z_k - \gamma^2 w_k^T w_k \right] \right\}$$

under zero initial condition, $V(x_k, i)|_{k=0} = 0$, and we have

$$J < \sum_{k=0}^{\infty} \left[z_k^T z_k - \gamma^2 w_k^T w_k + \Delta V(x_k, i) \right] = \sum_{k=0}^{\infty} \zeta_k^T \Phi_i^m \zeta_k$$

where $\zeta_k \triangleq \left[x_k^T \ w_k^T \right]^T$ and

$$\Phi_i^m \triangleq$$
$$\begin{bmatrix} A_i^T \tilde{\mathcal{P}}_i^m A_i - P_i + C_i^T C_i & A_i^T \tilde{\mathcal{P}}_i^m E_i + C_i^T F_i \\ * & -\gamma_i^2 I + E_i^T \tilde{\mathcal{P}}_i^m E_{i,s} + F_i^T F_i \end{bmatrix}$$

By Schur complement, (3) guarantees $\Phi_i^m < 0$, which means $J < 0$, i.e., $\|z\|_{E_2} < \gamma \|w\|_2$. Therefore, against the arbitrary variation, we conclude that system (1) is stochastically stable and has a prescribed H_∞ performance if (3) holds, which completes the proof. □

Remark 2. Note that (4) can be seen as the counterpart, in the discrete-time domain, of the stable condition derived in Boukas [2005]. Regardless the information of the variation of TP matrices, the approach may be probably conservative.

Taking the information on the variation of TP matrices into account, we can derive a less conservative H_∞ performance analysis criterion. In the following theorem, the BRL for system (1) scheduled based on the variation of TP matrices is given.

Theorem 2. Consider system (1) and let $0 < \alpha < 1$ and $\mu \geq 1$ be given constants. If there exist matrices $P_{i,s} > 0, \forall \sigma_k = s, s \in \mathcal{M}, r_k = i, i \in \mathcal{I}$, such that $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j$,

$$P_{i,s} < \mu P_{j,s} \tag{5}$$

$$\Psi_{i,s} \triangleq \begin{bmatrix} -\tilde{\mathcal{P}}_{i,s} & 0 & \tilde{\mathcal{P}}_{i,s} A_i & \tilde{\mathcal{P}}_{i,s} E_i \\ * & -I & C_i & F_i \\ * & * & -(1-\alpha) P_{i,s} & 0 \\ * & * & * & -\gamma_i^2 I \end{bmatrix} < 0, \tag{6}$$

with $\tilde{\mathcal{P}}_{i,s} \triangleq \sum_{j=1}^{N} \pi_{ij}^s P_{j,s}$, then the system is stochastically stable and has a guaranteed H_∞ performance index $\gamma_s = \max\{\gamma_i\}$ for any switching signal with ADT satisfying

$$\tau_a > \tau_a^* = -\ln \mu / \ln \bar{\alpha} \tag{7}$$

Proof. Construct a Lyapunov function as the following quadratic form:

$$V(x_k, r_k, \sigma_k) = x_k^T P(r_k, \sigma_k) x_k$$

Then, we have

$$\Delta V(x_k, i, s) = E[V(x_{k+1}, r_{k+1}, \sigma_k) | x, i, s] - V(x_k, i, s)$$
$$= x_{k+1}^T \left(\sum_{j=1}^{N} \pi_{ij}^s P_{j,s} \right) x_{k+1} - x_k^T P_{i,s} x_k$$
$$= x_k^T \left[A_i^T \tilde{\mathcal{P}}_{i,s} A_i - P_{i,s} \right] x_k$$
$$+ 2 x_k^T A_i^T \tilde{\mathcal{P}}_{i,s} E_i w_k + w_k^T E_i^T \tilde{\mathcal{P}}_{i,s} E_i w_k$$
$$\leq -\alpha x_k^T P_{i,s} x_k$$

Assuming $w_k \equiv 0$, adding up $\triangle V(x_k, i, s)$ from k_l to k and taking expectations, then by some mathematical operations, we have

$$E[V(x_{k+1}, r_{k+1}, s) | x_{k_l}, r_{k_l}, \sigma_{k_l}]$$
$$\leq (1-\alpha)^{(k-k_l)} E[V(x_{k_l}, r_{k_l}, \sigma_{k_l})] \tag{8}$$

Since (5) holds at switching instant k_l, we have

$$E[V x_{k_l}, r_{k_l}, \sigma_{k_l}] \leq \mu E[V x_{k_l}, r_{k_l}, \sigma_{k_{l-1}}] \tag{9}$$

Together with (8) and (9), one obtains

$$E[V(x_k, r_k, \sigma_k) | x_{k_0}, r_{k_0}, \sigma_{k_0}]$$
$$\leq (1-\alpha)^{(k-k_l)} \mu E[V(x_{k_l}, r_{k_l}, \sigma_{k_{l-1}}) | x_{k_0}, r_{k_0}, \sigma_{k_0}]$$
$$\leq \cdots$$
$$\leq (1-\alpha)^{(k-k_0)} \mu^{N_\sigma(k_0, k)} E[V x_{k_0}, r_{k_0}, \sigma_{k_0}]$$
$$\leq (1-\alpha)^{(k-k_0)} \mu^{N_0 + \frac{k-k_0}{\tau_a}} E[V(x_{k_0}, r_{k_0}, \sigma_{k_0})]$$
$$= \mu^{N_0} (1-\alpha)^{(k-k_0)} \mu^{\frac{k-k_0}{\tau_a}} E[V(x_{k_0}, r_{k_0}, \sigma_{k_0})]$$
$$= \mu^{N_0} e^{(k-k_0)\left[\ln(1-\alpha) + \frac{1}{\tau_a} \ln \mu\right]} E[V(x_{k_0}, r_{k_0}, \sigma_{k_0})] \tag{10}$$

If ADT satisfies (7), we have

$$\ln(1-\alpha) + \frac{1}{\tau_a} \ln \mu < 0$$

that is,

$$e^{\ln(1-\alpha) + \frac{1}{\tau_a} \ln \mu} < 1$$

Then, let $\varepsilon \triangleq e^{\ln(1-\alpha) + \frac{1}{\tau_a} \mu}$ and $c \triangleq \mu^{N_0}$, (10) is equal to

$$E[V(x_k, r_k, \sigma_k) | x_{k_0}, r_{k_0}, \sigma_{k_0}] \leq c\varepsilon^k E[V(x_{k_0}, r_{k_0}, \sigma_{k_0})]$$

Hence, we have

$$E\left[\sum_{k=0}^{N_\sigma(K,k)} V(x_k, r_k, \sigma_k) | x_{k_0}, r_{k_0}, \sigma_{k_0} \right]$$
$$\leq c \left(1 + \varepsilon + \cdots + \varepsilon^{N_\sigma(K,k)} \right) V(x_{k_0}, r_{k_0}, \sigma_{k_0})$$
$$= \frac{c \left(1 - \varepsilon^{N_\sigma(K,k)+1} \right) V(x_{k_0}, r_{k_0}, \sigma_{k_0})}{1 - \varepsilon}$$

that is,

$$\lim_{N_\sigma(K,k) \to \infty} E\left[\sum_{k=0}^{N_\sigma(K,k)} V(x_k, r_k, \sigma_k) | x_{k_0}, r_{k_0}, \sigma_{k_0} \right]$$
$$\leq \frac{cV(x_{k_0}, r_{k_0}, \sigma_{k_0})}{1 - \varepsilon}$$

Defining

$$M(x_{k_0}, r_{k_0}, \sigma_{k_0}) \triangleq \left(\min_{i \in \mathcal{I}, s \in \mathcal{M}} \lambda_{\min} P_{i,s} \right)^{-1} \frac{cV(x_{k_0}, r_{k_0}, \sigma_{k_0})}{1 - \varepsilon}$$

since

$$V(x_k, r_k, \sigma_k) = x_k^T P(r_k, \sigma_k) x_k$$
$$\geq x_k^T \left(\min_{i \in \mathcal{I}, s \in \mathcal{M}} \lambda_{\min} P_{i,s} \right) x_k$$

which means

$$\lim_{N_\sigma(K,k) \to \infty} E\left[\sum_{k=0}^{N_\sigma(K,k)} \|x(k)\|^2 | x_{k_0}, r_{k_0}, \sigma_{k_0} \right]$$
$$\leq M(x_{k_0}, r_{k_0}, \sigma_{k_0}) < \infty$$

then the system is stochastically stable.

Hence, if $\Delta V(x_k, i, s) + \alpha V(x_k, i, s) < 0$, combining with (5) and (7), we have $E\left\{\sum_{k=0}^{\infty} ||x_k||^2 |x_0, r_0\right\} < \infty$, that is, the system is stochastically stable.

On the other hand, by Schur complement, denoting $\Omega_{i,s} \triangleq A_i^T \tilde{\mathcal{P}}_{i,s} A_i - (1-\alpha) P_{i,s}$, (6) equals to

$$\begin{bmatrix} \Omega_{i,s} + C_i^T C_i & A_i^T \tilde{\mathcal{P}}_i E_i + C_i^T F_i \\ * & -\gamma_i^2 I + E_i^T \tilde{\mathcal{P}}_{i,s} E_i + F_i^T F_i \end{bmatrix} < 0 \quad (11)$$

It follows from (11) that

$$E\left\{\zeta^T \begin{bmatrix} \Omega_{i,s} + C_i^T C_i & A_i^T \tilde{\mathcal{P}}_{i,s} E_i + C_i^T F_i \\ * & -\gamma_i^2 I + E_i^T \tilde{\mathcal{P}}_{i,s} E_i + F_i^T F_i \end{bmatrix} \zeta\right\}$$
$$< 0$$

where $\zeta \triangleq \begin{bmatrix} x_k^T & w_k^T \end{bmatrix}^T$. Denoting $\Gamma(d) \triangleq z_d^T z_d - \gamma_i^2 w_d^T w_d$, we have

$$E\left[V(x_{k+1}, r_{k+1}, \sigma_k)|x, i, s\right]$$
$$\leq (1-\alpha) E(V(x_k, r_k, \sigma_k)) - E(\Gamma(k)) \quad (12)$$

Sum up (12) from k_0 to k and denote $\bar{\alpha} \triangleq 1 - \alpha$, yields $\forall \sigma_k = s, s \in \mathcal{M}$,

$$E\{V(x_k, r_k, s)|x_{k_0}, r_{k_0}, s\}$$
$$\leq \bar{\alpha}^{k-k_0} E\{V(x_{k_0}, r_{k_0}, s)\} - \sum_{d=k_0}^{k-1} \bar{\alpha}^{k-d-1} E(\Gamma(d))$$

One has $E\{V(x_{k_0}, r_{k_0}, s)\} = V(x_{k_0}, r_{k_0}, s) = 0$ and $E\{V(x_k, r_k, s)|x_{k_0}, r_{k_0}, s\} \geq 0$, under zero condition, thus

$$\sum_{d=k_0}^{k-1} \bar{\alpha}^{k-d-1} E(\Gamma(d)) \leq 0$$

Since $E(w_d^T w_d) = w_d^T w_d$, denoting $\breve{\alpha} \triangleq \bar{\alpha}^{k-d-1}$, we have

$$\sum_{d=k_0}^{k-1} \breve{\alpha} E(z_d^T z_d) \leq \sum_{d=k_0}^{k-1} \breve{\alpha} \gamma_i^2 w_d^T w_d$$

Therefore,

$$\sum_{k=k_0}^{\infty} \sum_{d=k_0}^{k-1} \breve{\alpha} E(z_d^T z_d) \leq \sum_{k=k_0}^{\infty} \sum_{d=k_0}^{k-1} \breve{\alpha} \gamma_i^2 w_d^T w_d$$
$$\Leftrightarrow \sum_{d=k_0}^{\infty} \sum_{k=d}^{\infty} \breve{\alpha} E(z_d^T z_d) \leq \sum_{d=k_0}^{\infty} \sum_{k=d}^{\infty} \breve{\alpha} \gamma_i^2 w_d^T w_d$$
$$\Leftrightarrow \sum_{d=k_0}^{\infty} \frac{1}{\alpha\bar{\alpha}} E(z_d^T z_d) \leq \sum_{d=k_0}^{\infty} \frac{1}{\alpha\bar{\alpha}} \gamma_i^2 w_d^T w_d$$
$$\Leftrightarrow \sum_{d=k_0}^{\infty} E(z_d^T z_d) \leq \sum_{s=k_0}^{\infty} \gamma_i^2 w_d^T w_d$$

As a result, for system (1) with s^{th} TP Matrix, we know the H_∞ performance index is not greater than γ_i. Therefore, we conclude that system (1) is stochastically stable for any switching signal satisfying (7) and has a guaranteed H_∞ performance index $\gamma_s = \max\{\gamma_i\}$. $\square$

4. A NUMERICAL EXAMPLE

In this section, a numerical example will be given to show the validity and potential of our developed theoretical results in the discrete-time case.

Example 1. Consider MJLS (1) with two operation modes and the following data:

$$A_1 = \begin{bmatrix} 0.88 & -0.05 \\ 0.4 & -0.72 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.8 & 0.16 \\ 0.8 & 0.64 \end{bmatrix},$$

$$C_1 = [\,0.2\ 0.1\,], \quad C_2 = [\,-0.1\ 0.2\,],$$

$$E_1 = [\,0.7\ 1.3\,], \quad E_2 = [\,-1.1\ 0.9\,],$$

$$F_1 = 0.3, \quad F_2 = -1.1$$

The piecewise-constant TP matrices are given as:
$$TP_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{bmatrix}, TP_2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.1 \end{bmatrix}$$

Our purpose here is to derive a BRL and find out admissible switching signals for the system to check whether the system is stochastically stable with an H_∞ disturbance attenuation performance. The data of H_∞ performance computed by using different approaches are listed in the following table.

Table 1. H_∞ performance indices

Methods	τ_a^*	Minimum γ_s
Proposition 1	–	7.9774
Theorem 1 $\mu = 1.01, \alpha = 0.01$	0.99	3.0211
Theorem 1 $\mu = 1.01, \alpha = 0.0025$	3.9752	2.9582

In the above example, it can be seen from the comparison in Table 1 that, with further modeling of Markov process for the variation of TR or TP matrices, the improvement of Theorem 1 over Proposition 1 in concern of conservatism is quite obvious. This is resulted from the fact that Proposition 1 without extra knowledge on the variation of the TP matrices has to consider all the TPs. In addition, it is easily observed that the slower the TP matrices vary, the less conservative H_∞ performance index can be achieved.

5. CONCLUSIONS

The H_∞ performance analysis problem for a class of discrete-time MJLSs with piecewise-constant TPs is investigated. The variations on the piecewise-constant TPs are subject to ADT switching. Utilizing a special construction of Lyapuov function, a BRL for the underlying system is derived in order to check whether the system is stochastically stable and has a guaranteed H_∞ noise-attenuation performance index scheduled based on the variation of TP matrices. A numerical example demonstrates the theoretical finding. The idea and method behind this paper can be thereby used to deal with the other problems of the underlying system such as H_∞ control, H_∞ estimation, H_∞ model reduction, etc.

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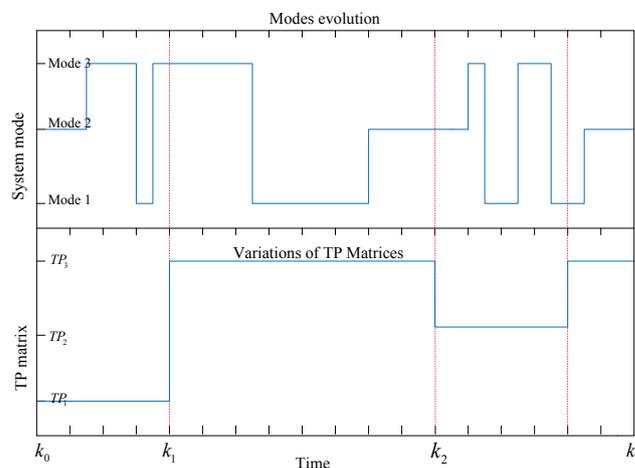
Fig. 1. A variation of TP matrices subject to deterministic switching signal and the corresponding possible path of modes evolution