Asynchronous Observer Design for Switched Linear Systems: A Tube-Based Approach

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***Abstract*— This paper proposes a tube-based method for the asynchronous observation problem of discrete-time switched linear systems in the presence of amplitude-bounded disturbances. Sufficient stability conditions of the nominal observer error system under mode-dependent persistent dwell-time (MPDT) switching are first established. Taking the disturbances into account, a novel asynchronous MPDT robust positive invariant (RPI) set and an asynchronous MPDT generalized RPI (GRPI) set are determined for the difference system between the nominal and disturbed observer error systems. Further, the global uniform asymptotical stability of the observer error system is established in the sense of converging to the asynchronous MPDT GRPI set, i.e., the cross section of the tube of the observer error system. Finally, the proposed results are validated on a space robot manipulator example.**

***Index Terms*—Asynchronous observer design, generalized robust positive invariant (GRPI) set, mode-dependent persistent dwell-time (MPDT), switched linear systems.**

I. Introduction

THE asynchronous phenomenon in switched systems commonly results from delays caused by detecting mode switchings as well as designing new controllers and observers for unknown modes at runtime. As shown in [1], the master-slave coordination system often switches among high-gain and low-gain controllers when the slave changes from being in a contact-free motion to interacting with a stiff environment, or vice versa. The inevitable delay caused by this controller switching is detrimental to system stability and performance. Another example can be found in the framework of adaption and learning via multiple models switching [2], where back-up models are needed to deal with unpredictable changes of the environment on top of the predefined models. Similar to other typical time-delay systems, time delays in the asynchronous switching may also cause performance degradation and even system instability, as shown in the studies of different topics of such systems, [3]–[15], to name a few.

A fundamental problem in the area of asynchronous switched systems is disturbance handling, and the difficulties largely depend on the types of the disturbances. For the systems with energy-bounded ones, i.e., l_2 norms, results have been well-established in the literature [16], [17]. Note that the l_∞ disturbances are common in many practical systems, such as valve control systems [18], robotic control systems [19], aircraft flight control systems and shipping navigation control systems [20]. To our best knowledge, the amplitude-bounded disturbances, i.e., l_∞ norms, have not been investigated for the asynchronous switched systems yet. An effective way to handle l_∞ disturbances is the tube based method, for example, in stabilization [21] and advanced model predictive control [22].

The crux of applying the tube-based method to the l_∞ disturbance rejection problem for switched systems is to guarantee that the error system states remain within certain formally-defined robust sets. To this end, a robust positive invariant (RPI) set is determined such that the state trajectory of the error system always remains within the RPI set at switching instants while staying within an outer robust set during subsystem evolvement [23]. This outer robust set is named as the generalized robust positive invariant (GRPI) set and determined based on the RPI set. Then, by shifting the center of the GRPI set from the origin to the nominal trajectory of the switched system at each instant, a tube that contains all possible state trajectories is constructed and stability condition is established accordingly. Results determining the RPI set can be found in the cases of persistent dwell-time (PDT) [21], dwell-time (DT) [23], and average dwell-time (ADT) switching [24]. In the context of asynchronous switching, the aforementioned procedures would become more challenging than in the synchronous case, due to the complicated computation of RPI and GRPI sets.

Motivated by the observations above, this study focuses on the asynchronous observation problem for discrete-time switched linear systems with amplitude-bounded additive disturbances. The switching signals pertain to a class of mode-dependent persistent dwell-time (MPDT) switching and the disturbance is considered to be l_∞ finite. The contributions of this paper lie in that: 1) Sufficient stability conditions of the

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nominal observer error system under the asynchronous MPDT switching are proposed and an algorithm is designed to determine the asynchronous observer solution. 2) The RPI and GRPI sets in the asynchronous MPDT switching case are first determined in this paper and the corresponding algorithm is designed. 3) Based on the determined asynchronous MPDT GRPI set, the stability condition of the disturbed observer error system is obtained.

The remainder of this paper is organized as follows. In Section II, the problem formulation is presented and basic concepts are given. The detailed derivations of the proposed results are given in Section III. An application of the obtained results to a space robot manipulator is given in Section IV, and Section V concludes this paper.

Notations: In this paper, $\mathbb{R}^n$ refers to the n dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the sets of integers and non-negative integers respectively; $\mathbb{Z}_{\geq a}$ and $\mathbb{Z}_{[a,b]}$ denote the sets $\{k\in\mathbb{Z}|k\geq a\}$ and $\{k\in\mathbb{Z}|a\leq k\leq b\}$, respectively, $0\leq a\leq b$. The Minkowski sum and Pontryagin difference of two compact sets, $\Theta_1\subseteq\mathbb{R}^n$ and $\Theta_2\subseteq\mathbb{R}^n$, are $\Theta_1\oplus\Theta_2=\{x_1+x_2|x_1\in\Theta_1,x_2\in\Theta_2\}$ and $\Theta_1\ominus\Theta_2=\{x|x+x_2\in\Theta_1,x_2\in\Theta_2\}$, respectively. $co\{\cdot\}$ denotes the convex hull of a set. Let $\mathcal{B}^n$ denote a unit ball set $\{x\in\mathbb{R}^n|\,\|x\|_2\leq 1\}$. A function $\kappa:[0,\infty)\to[0,\infty)$ is an $\mathcal{K}_\infty$ class function if it is strictly increasing, continuous, unbounded and $\kappa(0)=0$. For a vector $x\in\mathbb{R}^n$ and a set $\Theta\subseteq\mathbb{R}^n$, the distance between x and Θ is defined as $\|x\|_\Theta=\inf_{y\in\Theta}\|x-y\|$. $\mathbb{S}^n_{>0}$ ($\mathbb{S}^n_{\geq 0}$) denotes the set of $n\times n$ symmetric positive (semi-positive) definite matrices. In addition, diag$\{X,Y\}$ stands for a block-diagonal matrix where diagonal entries are X and Y. Symbol $*$ is used as an ellipsis for the terms that are introduced by symmetry. I and 0 represent the identity matrix and zero matrix, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. $\prod_{l=N_1}^{N_2}A_l$ stands for $A_{N_2}A_{N_2-1}\cdots A_{N_1}$, $N_1<N_2$. $\bigcup_{i\in\mathcal{I}}\Theta_i$ stands for $\Theta_N\bigcup\Theta_{N-1}\bigcup\cdots\bigcup\Theta_1$, $\Theta_i\subseteq\mathbb{R}^n$, $\forall i\in\mathcal{I}=\{1,2,\dots,N\}$.

II. Preliminaries and Problem Formulation

Consider the discrete-time switched linear systems and Luenberger observer described as follows:

$$\begin{cases}x_{k+1}=A_{\sigma(k)}x_k+B_{\sigma(k)}w_k\\ y_k=C_{\sigma(k)}x_k\end{cases}\tag{1}$$

$$\begin{cases}\widehat{x}_{k+1}=A_{\sigma(k)}\widehat{x}_k+L_{\sigma(k)}(\varphi_k)(y_k-\widehat{y}_k)\\ \widehat{y}_k=C_{\sigma(k)}\widehat{x}_k\end{cases}\tag{2}$$

where $x_k\in\mathbb{R}^{n_x}$ and $\hat{x}_k\in\mathbb{R}^{n_x}$ are the states of the system and observer, respectively; $y_k\in\mathbb{R}^{n_y}$ and $\hat{y}_k\in\mathbb{R}^{n_y}$ are the outputs of the system and observer, respectively; $w_k\in\mathbb{W}\subseteq\mathbb{R}^{n_u}$ is the amplitude-bounded additive disturbance which includes both exogenous disturbance and the modelling error [25], [26], and $\mathbb{W}$ is a compact polyhedral set which contains the origin in its interior. The switched linear system (1) consists of several subsystems and the switching between different subsystems is governed by the switching signal $\sigma(k)$, which takes value in a finite set $\mathcal{I}=\{1,2,\dots,N\}$, where N is the number of subsystems. In this paper, a quasi-time dependent (QTD) Lyapunov function is adopted similar to the one proposed in [17], upon which a QTD Luenberger observer (2) is designed such that the observer error system is globally uniformly asymptotically stable, where $L_{\sigma(k)}(\varphi_k)$ is the QTD observer gain and φ_k is a scheduled index for the activated subsystem. In addition, the non-QTD observer can be obtained by setting $L_{\sigma(k)}(\varphi_k)\equiv L_{\sigma(k)}$, $\forall\varphi_k\in\mathbb{Z}_{\geq 0}$.

In this paper, $\sigma(k)$ is considered to belong to the set of mode-dependent persistent dwell-time (MPDT) switching. The concept of MPDT is given in the following definitions.

Definition 1 [27]: Consider system (1) and switching instants $k_0,k_1,\dots,k_s,\dots$ with $k_0=0$. A positive constant τ (respectively τ_i) is 1) the dwell-time if for all $k\geq 0$, $k_{s+1}-k_s\geq\tau$; 2) the mode-dependent dwell-time of the ith mode of system (1) if for all $k\geq 0$ such that $\sigma(k)=i$ for $k\in[k_s,k_{s+1})$, $k_{s+1}-k_s\geq\tau_i$.

Definition 2 [21]: Consider system (1) and switching instants $k_0,k_1,\dots,k_s,\dots$ with $k_0=0$. If there exist infinite disjoint intervals of length no smaller than τ_i on which $\sigma=i$, and consecutive intervals with the same property are separated by no more than T, then τ_i is called the mode-dependent persistent dwell-time and T is called the period of persistence.

The switching sequence satisfying Definition 2 is coined as an MPDT switching sequence. Then, $[\tau]^T:=\{\tau_1,\tau_2,\dots,\tau_N\}$ denotes the set of MPDT τ_i's and $\zeta_{[\tau]^T}(k)$ is the set of all admissible MPDT switching sequences till time k.

As illustrated in Fig. 1, the interval composed of a dwell-time portion and a persistence portion can be regarded as an MPDT phase [21]. k_s is the initial instant of the sth phase as well as the instant switching into the sth phase. In the dwell-time portion, one subsystem $\Omega_{\sigma(k_s)}$ is maintained for at least $\tau_{\sigma(k_s)}$. Whereas in the persistence portion, more than one switchings happen and each subsystem sustains for the time duration of $T_{\sigma(k_s^r)}$, $T_{\sigma(k_s^r)}<\tau_{\sigma(k_s^r)}$, where k_s^r is the switching instant between k_s and k_{s+1}, $k_s^r\in\mathbb{Z}_{[k_s,k_{s+1}]}$. $T^{(s)}$ is the running time of the persistence portion in the sth phase,

$$T^{(s)}:=\sum_{r=1}^{Q(k_s,k_{s+1})}T_{\sigma(k_s^r)}\leq T$$

where $Q(k_a,k_b)$ denotes the number of switchings between two instants, a and $b\in\mathbb{Z}$.

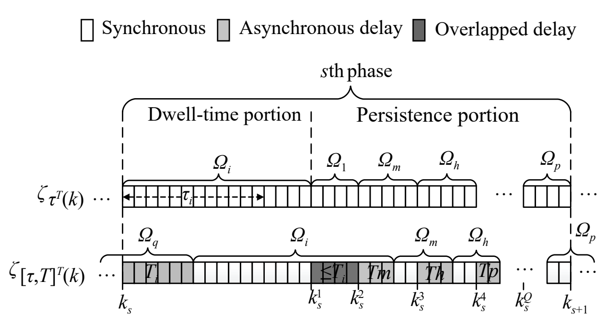


Fig. 1. General illustration of an MPDT switching sequence and an asynchronous MPDT switching sequence.

In the proposed observer (2), the QTD observer gain $L_{\sigma(k)}(\varphi_k)$ is scheduled by the index φ_k, which is computed online as in [17]. $\forall\sigma(k)=i\in\mathcal{I}$,

1) in the dwell-time portion

$$\varphi_k = \begin{cases} k-k_s, & k\in[k_s, k_s+\tau_i) \\ \tau_i, & k\in\left[k_s+\tau_i, k_s^1\right) \end{cases}$$

2) in the persistence portion

$$\varphi_k = k-H_r \; k\in\left[k_s^1, k_{s+1}\right)$$

where $H_r = \arg\max(k_s^r|\sigma(k_s^r)\neq\sigma(k_s^r+1), k_s^r\leq k)$, $r\in\mathbb{Z}_{[0,Q(k_s,k_{s+1})]}$ and $H_0 = k_s$.

In the case of synchronous observation, the observer always switches simultaneously with the switched linear system (1); however, this doesn not hold in most practical applications [28]. Consider the case where a new subsystem is identified at runtime, and the observer gains need to be recalculated online accordingly. In this case, if the switched linear system (1) consists of numerous high-order subsystems, solving the observer design problem online could take more than one sampling period, likely causing asynchronous switching between the system modes and observer gains. In the presence of asynchronous delays, the observer becomes

$$\begin{cases} \widehat{x}_{k+1} = A_{\sigma(k)}\widehat{x}_k + L_{\hat{\sigma}(k)}(\varphi_k)(y_k-\widehat{y}_k) \\ \widehat{y}_k = C_{\sigma(k)}\widehat{x}_k \end{cases} \tag{3}$$

where $\hat{\sigma}(k)$ is the asynchronous MPDT switching signal, governing the switching of observer gains and taking value in the same set with $\sigma(k)$.

The asynchronous delay caused by the identification of switching and online computation of observer gains for the new subsystem is denoted by $\mathcal{T}$. Let $\mathcal{T}_\uparrow(a,b)$ and $\mathcal{T}_\downarrow(a,b)$ denote the set of unmatched and matched periods in $[a,b)$, respectively. Similar to τ, $\mathcal{T}$ is also mode-dependent and the set of asynchronous mode-dependent persistent dwell-time is denoted by $[\tau,\mathcal{T}]^T := \{(\tau_i,\mathcal{T}_i), i\in\mathcal{I}\}$. Given the MPDT switching signal $\sigma(k)$, the observer switching signal $\hat{\sigma}(k)$ can be derived according to the following rule:

$$\hat{\sigma}(k) = \begin{cases} \hat{\sigma}(k-1), & k\in[H_r, H_r+\mathcal{T}_{\sigma(H_r)}) \\ \sigma(H_r), & k\in[H_r+\mathcal{T}_{\sigma(H_r)}, H_{r+1}). \end{cases}$$

The switching sequence composing of $\hat{\sigma}(k)$ is called an asynchronous MPDT switching sequence. The set of all admissible asynchronous MPDT switching sequences till k is denoted by $\zeta_{[\tau,\mathcal{T}]^T}(k)$.

The relation between $\sigma(k)$ and $\hat{\sigma}(k)$ is illustrated in Fig. 1. In the dwell-time portion, only one switching occurs and waits for at least τ_i moments, causing only one period of asynchronous delay (see $\mathcal{T}_i$). However in the persistence portion, more than one switching could occur. The asynchronous delays invoked by fast switchings may be overlapped (i.e., a new switching happens before the end of the previous delay, see $\mathcal{T}_i$ and $\mathcal{T}_m$) or cover the entire persistence portion (i.e., new switchings happen succeedingly before or at the end of previous delays).

Fig. 2 provides an illustration of how the proposed observer is executed in practice. This paper is concerned with the observer design and as such the corresponding parts therein connected by the blue arrows, an observer-based controller is also shown for the readers to have a comprehensive view of the close-looped system. In the space robot manipulator example in the sequel, the observer-based controller will be used to only maintain the system stability, but will not be discussed in detail in the main results.

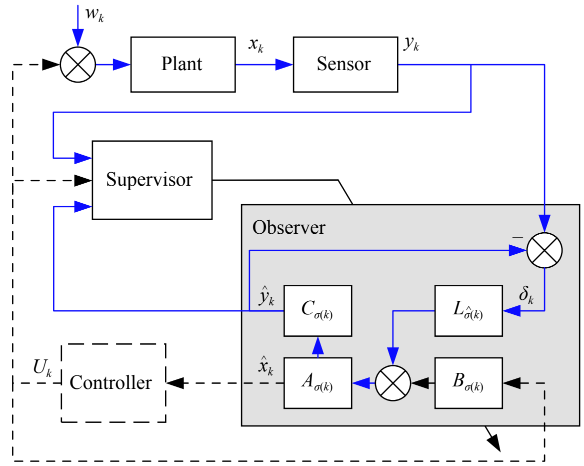


Fig. 2. Illustration of the observer structure and signal flow. The blue lines show the closed-loop observed system structure, and the dashed lines are used to show the structure when the observer-based controller is involved.

Let $\varepsilon_k := x_k-\widehat{x}_k$, the resulting disturbed observer error system between systems (1) and (3) becomes

$$\varepsilon_{k+1} = \begin{cases} A_i\varepsilon_k - L_j(\varphi_k)C_i\varepsilon_k + B_iw_k, & k\in\mathcal{T}_\uparrow(k_s,k_{s+1}) \\ A_i\varepsilon_k - L_i(\varphi_k)C_i\varepsilon_k + B_iw_k, & k\in\mathcal{T}_\downarrow(k_s,k_{s+1}) \end{cases} \tag{4}$$

where $\sigma(k) = i\neq j = \sigma(k_s-1)$, $\forall(i,j)\in\mathcal{I}\times\mathcal{I}$. Let $w\equiv 0$, the nominal observer error system becomes

$$z_{k+1} = \begin{cases} A_iz_k - L_j(\varphi_k)C_iz_k, & k\in\mathcal{T}_\uparrow(k_s,k_{s+1}) \\ A_iz_k - L_i(\varphi_k)C_iz_k, & k\in\mathcal{T}_\downarrow(k_s,k_{s+1}). \end{cases} \tag{5}$$

Let $e_k := \varepsilon_k - z_k$, the *difference system* between the disturbed observer error system (4) and the nominal system (5) becomes

$$e_{k+1} = \begin{cases} \bar{A}_{i,j}(\varphi_k)e_k + B_iw_k, & k\in\mathcal{T}_\uparrow(k_s,k_{s+1}) \\ \bar{A}_{i,i}(\varphi_k)e_k + B_iw_k, & k\in\mathcal{T}_\downarrow(k_s,k_{s+1}) \end{cases} \tag{6}$$

where $\bar{A}_{i,j}(\varphi_k) = A_i - L_j(\varphi_k)C_i$, $\forall(i,j)\in\mathcal{I}\times\mathcal{I}$.

To establish the stability criterion for the systems above, we need the following definitions:

Definition 3 [29]: A set $O\subset\mathbb{R}^{n_x}$ is said to be a robust positive invariant (RPI) set for system $x_{k+1} = f(x_k,w_k)$, $w_k\in\mathbb{W}$, if $x_k\in O$ implies $x_t\in O$ for any $w_t\in\mathbb{W}$, $t\in\mathbb{Z}_{\geq k+1}$.

Definition 4: A set $O([\tau,\mathcal{T}]^T)$ is said to be an asynchronous MPDT RPI set for system (6) with asynchronous MPDT $[\tau,\mathcal{T}]^T$, if $e_{k_0}\in O([\tau,\mathcal{T}]^T)$ implies $e_{k_s}\in O([\tau,\mathcal{T}]^T)$ for every admissible switching in $\zeta_{[\tau,\mathcal{T}]^T}(t)$ with $w_t\in\mathbb{W}$, $t\in\mathbb{Z}_{\geq k_0}$, $s\in\mathbb{Z}_{\geq 1}$.

Definition 5: A set $\mathcal{G}([\tau,\mathcal{T}]^T)\subseteq\mathbb{R}^{n_x}$ is said to be an asynchronous MPDT generalized robust positive invariant (GRPI) set for system (6) with asynchronous MPDT set $[\tau,\mathcal{T}]^T$, if $e_k\in O([\tau,\mathcal{T}]^T)\subseteq\mathcal{G}([\tau,\mathcal{T}]^T)$ implies $e_t\in\mathcal{G}([\tau,\mathcal{T}]^T)$ for every admissible switching in $\zeta_{[\tau,\mathcal{T}]^T}(t)$ and for any $w_t\in\mathbb{W}$, $t\in\mathbb{Z}_{\geq k+1}$, where $O([\tau,\mathcal{T}]^T)$ is an asynchronous MPDT RPI set for system (6).

Definition 6 [30]: System (5) is globally uniformly asymptotically stable (GUAS) under certain switching signals σ if for initial condition z_{k_0}, there exists a class of $\mathcal{K}_\infty$ function

κ such that the solution of the system satisfies $\|z_k\| \le \kappa(\|z_0\|)$, $\forall k \in \mathbb{Z}_{\ge k_0}$ and $\|z_k\| \to 0$ as $k \to \infty$.

Definition 7: An asynchronous MPDT GRPI set $\mathcal{G}([\tau,\mathcal{T}]^T) \subseteq \mathbb{R}^{n_x}$ is said to be GUAS for system (6) with asynchronous MPDT switching, if for all $k \in \mathbb{Z}_+$, $\|\varepsilon_k\|_{\mathcal{G}([\tau,\mathcal{T}]^T)} \le \kappa(\|\varepsilon_0\|_{\mathcal{G}([\tau,\mathcal{T}]^T)})$ and $\|\varepsilon_k\|_{\mathcal{G}([\tau,\mathcal{T}]^T)} \to 0$ as $k \to \infty$, where $\kappa \in \mathcal{K}_\infty$.

Considering all the asynchronous phenomena shown in Fig. 1, this paper aims to design a full-order state observer for the switched linear system (1) subject to asynchronous MPDT switching regularities, and find an asynchronous MPDT GRPI set for the resulting difference system (6) such that the disturbed observer error system (4) is GUAS in the sense of Definition 7.

III. Main Results

In this section, we first investigate the stability criteria and the QTD observer design for the nominal observer error system (5) in the presence of asynchronous delay, under MPDT switching. Subsequently, the asynchronous MPDT RPI set and asynchronous MPDT GRPI set of the difference system (6) are determined. Finally, the stability of the disturbed observer error system (4) is established in the sense of Definition 7. The relations between the criteria obtained in this section are illustrated in Fig. 7 in Appendix A. Most of the proofs in the first subsection can also be found in Appendix B.

A. Nominal Systems

A class of QTD Lyapunov functions $V_i(x_k,k)$, allowing the energy to increase when the unmatched observer is activated, are considered to establish the stability criterion for the nominal system in nonlinear cases. The energy increment should be compensated by the decrement during matched stages, such that the overall system is stable. To this end, the increasing and declining rates are restricted below certain values in the following lemma.

Lemma 1: Consider a discrete-time switched system $x_{k+1} = f_{\sigma(k)}(x_k)$, $0 < \alpha < 1$, $\beta > 0$ and $\mu \ge 1$ are given constants. For a prescribed period of persistence $\mathcal{T}$, suppose that there exist a family of functions $V_{\sigma(k)} : (\mathbb{R}^{n_x}, \mathbb{Z}_{[0,\tau_{\sigma(k)}]}) \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions κ_1 and κ_2 such that: $\forall \sigma(k) = i \in \mathcal{I}$, $\forall k \in [k_s, k_s^1)$,

$$\kappa_1(\|x_k\|) \le V_i(x_k,\varphi_k) \le \kappa_2(\|x_k\|) \tag{7}$$

$\forall k \in [k_s, k_s + \mathcal{T}_i) \cup [k_s^r, k_s^r + \mathcal{T}_i)$,

$$V_i(x_{k+1},\varphi_k+1) \le \beta V_i(x_k,\varphi_k) \tag{8}$$

$\forall k \in [k_s + \mathcal{T}_i, k_s + \tau_i) \cup [k_s^r + \mathcal{T}_i, k_s^{r+1})$,

$$V_i(x_{k+1},\varphi_k+1) \le \alpha V_i(x_k,\varphi_k) \tag{9}$$

$\forall k \in [k_s + \tau_i, k_s^1)$,

$$V_i(x_{k+1},\tau_i) \le \alpha V_i(x_k,\tau_i) \tag{10}$$

$\forall (i,j) \in \mathcal{I} \times \mathcal{I}$ and $\forall r \in \mathbb{Z}_{[1,Q(k_s^1,k_{s+1})]}$,

$$V_i(x_{k_s^1},0) \le \mu V_j(x_{k_s^1},\tau_j) \tag{11}$$

$$V_i(x_{k_s^r},0) \le \mu V_j(x_{k_s^r},T_j) \tag{12}$$

where $T_j \in [1, \min(\tau_j - 1, T^{(s)})]$, $j \in \mathcal{I}$, $T^{(s)} \in \mathbb{Z}_{[0,T]}$.

Then the switched system is GUAS for asynchronous MPDT switching signals satisfying

$$\lambda_i = \mu^{T+1} \beta^T \alpha^{\tau_i - \mathcal{T}_i} \beta^{\mathcal{T}_i} \le 1, \forall i \in \mathcal{I} \tag{13}$$

Proof: See Appendix B. ■

Remark 1: It is worth noting that λ_i denotes the worst-case energy coefficient rate during a phase, when the system works asynchronously during the period of T in the persistence portion, resulting in the Lyapunov function to be increasing to the greatest extent. Thus, the system is guaranteed to be stable in the presence of consecutive asynchronous switchings as well as the overlapped asynchronous delays, as displayed in Fig. 1.

Next, the quasi-time-dependent (QTD) Lyapunov function is defined as $V_i(x_k,k) := x_k^T P_i(k) x_k$, and the stability criterion for the nominal observer error system (5) is established in the following theorem.

Theorem 1: Consider system (5), suppose there exist constants $0 < \alpha < 1$, $\beta \ge 1$, $\mu \ge 1$ and matrices $P_i(\varphi) \in \mathbb{S}_{>0}^{n_x}$ and $U_i(\varphi) \in \mathbb{R}^{n_x \times n_y}$, such that $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$,

$$\Phi_{i,j}(\varphi+1,\varphi,k) \le 0, \forall \varphi \in \mathbb{Z}_{[0,\mathcal{T}_i)}, \forall k \in \mathbb{Z}_{[\mathcal{T}_j,\tau_j)} \tag{14}$$

$$\Upsilon_i(\varphi+1,\varphi) \le 0, \forall \varphi \in \mathbb{Z}_{[\mathcal{T}_i,\tau_i)} \tag{15}$$

$$\Psi_i(\tau_i) \le 0 \tag{16}$$

$$P_i(0) - \mu P_j(\tau_i) \le 0 \tag{17}$$

$$P_i(0) - \mu P_j\left(T_j\right) \le 0 \tag{18}$$

hold, where $T_j \in \mathbb{Z}_{[1,\min(\tau_j-1,T^{(s)}))}$, $T^{(s)} \in \mathbb{Z}_{[0,T]}$ and

$$\Phi_{i,j}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} P_i(\theta_1) - 2P_j(\theta_3) & P_j(\theta_3)A_i - U_j(\theta_3)C_i \\ * & -\beta P_i(\theta_2) \end{bmatrix} \tag{19}$$

$$\Upsilon_i(\theta_1,\theta_2) = \begin{bmatrix} P_i(\theta_1) - 2P_i(\theta_2) & P_i(\theta_2)A_i - U_i(\theta_2)C_i \\ * & -\alpha P_i(\theta_2) \end{bmatrix} \tag{20}$$

$$\Psi_i(\theta) = \begin{bmatrix} -P_i(\theta) & P_i(\theta)A_i - U_i(\theta)C_i \\ * & -\alpha P_i(\theta) \end{bmatrix}. \tag{21}$$

Then the system (5) is GUAS for asynchronous MPDT signals satisfying

$$\tau_{\min} \ge -\frac{((T+\mathcal{T}_{\max})\ln\beta + (T+1)\ln\mu + \mathcal{T}_{\max}\ln\alpha)}{\ln\alpha} \tag{22}$$

where $\tau_{\min} = \min_{i\in\mathcal{I}} \tau_i$, $\mathcal{T}_{\max} = \max_{i\in\mathcal{I}} \mathcal{T}_i$. Moreover, the QTD observer gain is given by $L_i(\varphi) = P_i^{-1}(\varphi)U_i(\varphi)$, $\varphi \in \mathbb{Z}_{[\mathcal{T}_i,\tau_i]}$.

Proof: See Appendix C. ■

Remark 2: Since the asynchronous switching could occur between any two subsystems at unpredictable instants, all possible combinations of time schedules and subscripts in inequalities (14) are involved to restrict the energy increment caused by the asynchronous switching.

Remark 3: For given constants α, β, μ, the inequalities in Theorem 1 are linear matrix inequalities (LMI). The subminimal asynchronous MPDT can be obtained through bisection on these constants while guaranteeing a feasible solution for the LMIs.

A noteworthy fact is that if letting $\mathcal{T}_i = 0$, $\forall i \in \mathcal{I}$, the observer given in Theorem 1 will correspond to the case of synchronous observation. In this case, the synchronous nominal observer error system is obtained by setting $\mathcal{T}_{\uparrow}(k_s, k_{s+1}) = \emptyset$ and $\mathcal{T}_{\downarrow}(k_s, k_{s+1}) = [k_s, k_{s+1})$ in (5)

$$z_{k+1} = A_i z_k - L_i(\varphi_k) C_i z_k,\ k \in [k_s, k_{s+1}). \tag{23}$$

To compare with the performance of synchronous observers under asynchronous switching, we also present the QTD synchronous observer design as below. The proof can be obtained by setting $\mathcal{T}_i = 0$, $\forall i \in \mathcal{I}$ in the proof of Theorem 1 and thus is omitted here.

Corollary 1: Consider system (23), suppose there exist constants $0 < \alpha < 1$, $\mu \geq 1$ and matrices $P_i(\varphi) \in \mathbb{S}^{n_x}_{>0}$ and $U_i(\varphi) \in \mathbb{R}^{n_x \times n_y}$, such that $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$\Upsilon_i(\varphi + 1, \varphi) \leq 0, \forall \varphi \in \mathbb{Z}_{[0,\tau_i)} \tag{24}$$

$$\Psi_i(\tau_i) \leq 0 \tag{25}$$

$$P_i(0) - \mu P_j(\tau_i) \leq 0 \tag{26}$$

$$P_i(0) - \mu P_j\left(T_j\right) \leq 0 \tag{27}$$

hold, where $\Upsilon_i(\varphi+1, \varphi)$ and $\Psi_i(\tau_i)$ are defined in (20) and (21), respectively, $T_j \in \mathbb{Z}_{[1,\min(\tau_j - 1, T^{(s)}))}$, $T^{(s)} \in \mathbb{Z}_{[0,T]}$. Then the system (23) is GUAS for MPDT signals satisfying

$$\tau_{\min} \geq -\frac{((T+1)\ln\mu + \mathcal{T}_{\max}\ln\alpha)}{\ln\alpha} \tag{28}$$

where $\tau_{\min} = \min_{i \in \mathcal{I}} \tau_i$. Moreover, the synchronous QTD observer gain is given by $L_i(\varphi) = P_i^{-1}(\varphi) U_i(\varphi)$, $\varphi \in \mathbb{Z}_{[0,\tau_i]}$.

Although the QTD observer formulation presented in Theorem 1 is generally less conservative compared to the non-QTD ones [17], the resulting quasi-time-varying close-looped state transition matrix could impede the calculation of invariant sets in the next section. Thus, the non-QTD observer design is presented in the following corollary based on the φ_k-independent Lyapunov function $V_i(x_k) := x_k^T P_i x_k$, the proof can be obtained in a similar vein to the one of Theorem 1 and thus is omitted.

Corollary 2: Consider system (5), suppose there exist constants $0 < \alpha < 1$, $\beta \geq 1$, $\mu \geq 1$ and matrices $P_i \in \mathbb{S}^{n_x}_{>0}$ and $U_i \in \mathbb{R}^{n_x \times n_y}$, such that $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$,

$$\Phi_{i,j} \leq 0 \tag{29}$$

$$\Psi_i \leq 0 \tag{30}$$

$$P_i - \mu P_j \leq 0 \tag{31}$$

hold, where

$$\Phi_{i,j} = \begin{bmatrix} P_i - 2P_j & P_j A_i - U_j C_i \\ * & -\beta P_i \end{bmatrix} \tag{32}$$

$$\Psi_i = \begin{bmatrix} -P_i & P_i A_i - U_i C_i \\ * & -\alpha P_i \end{bmatrix}. \tag{33}$$

Then the system (5) is GUAS for asynchronous MPDT switching signals satisfying

$$\tau_{\min} \geq -\frac{((T + \mathcal{T}_{\max})\ln\beta + (T+1)\ln\mu + \mathcal{T}_{\max}\ln\alpha)}{\ln\alpha} \tag{34}$$

where $\tau_{\min} = \min_{i \in \mathcal{I}} \tau_i$, $\mathcal{T}_{\max} = \max_{i \in \mathcal{I}} \mathcal{T}_i$. Moreover, the non-QTD observer gain is given by

$$L_i = P_i^{-1} U_i,\ \forall i \in \mathcal{I} \tag{35}$$

Remark 4: The feasible solutions of Corollary 1 and Corollary 2 can be obtained following a similar procedure to Algorithm 1 by replacing the cost functions with (28) and (34), the constraints with (24)–(27) and (29)–(31), respectively.

Next, a numerical example is introduced to illustrate the validity of theoretical results in this section, as well as the discussions above.

Example 1: Consider a switched linear system with two subsystems

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.36 & 0.66 \\ -0.39 & 1.45 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.38 & 0.74 \\ 1.95 & 2.12 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 0.5 & 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}. \end{aligned} \tag{36}$$

Our aim is to design a QTD observer based on Theorem 1 and Corollary 1 for the system (36). To illustrate the effectiveness of the proposed observer design, a fairly non-conservative asynchronous MPDT set $[\tau, \mathcal{T}]^T$ is constrained as $\tau_{\min} = 8$, $\mathcal{T}_1 = \mathcal{T}_2 = 4$, $T^{(s)} \leq 6$. Since $\mathcal{T}_i$ is involved in τ_i, it is possible that the system runs in the unmatched condition for the greater part of the time. For the observer designed by Theorem 1, the constants searched through bisection are $\alpha = 0.5$, $\beta = 1.05$, $\mu = 1.05$. For Corollary 1, $\alpha = 0.3$ and $\mu = 1.05$.

Due to the fact that the subsystems of (36) are both unstable, a non-switching observer-based state-feedback controller is designed to protect the closed-loop system from divergence and the controller gain is obtained as $K = [-2.259, -4.239]$.

As demonstrated in Fig. 3, the asynchronous delays occupy more than half of the domain (see the bottom). When the

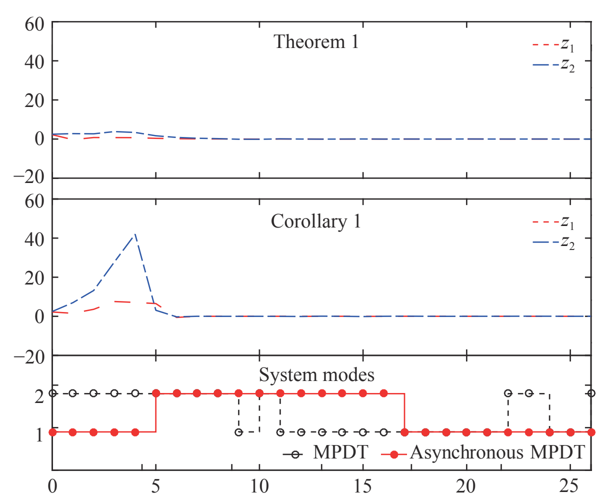


Fig. 3. State responses of the nominal observer error system (36), with observers designed by Theorem 1 and Corollary 1 under asynchronous MPDT switching.

synchronous observers are applied, the state response tends to become unstable and an overshoot appears in the presence of asynchronous switchings, while the asynchronous observers designed by Theorem 1 stabilizes the nominal observer error system of system (36) effectively.

B. Systems With Bounded Additive Disturbances

In this subsection, to address the stability of the system (6) in the sense of Definition 7, we will first determine an asynchronous MPDT RPI set and an asynchronous MPDT GRPI set. The GRPI set is developed on the basis of an RPI set, thus the form and existence proof of the RPI set need to be presented first. Given the observer gains determined by Corollary 1 as (35), the system (5) is GUAS under asynchronous MPDT switching. In the following theorem, the existence of an asynchronous MPDT RPI set is proved.

Theorem 2: If system (5) is GUAS with asynchronous MPDT $[\tau,\mathcal{T}]^T$, then there exists an asynchronous MPDT RPI set $O([\tau,\mathcal{T}]^T)$ for system (6).

Proof: First, define a set $S_{t(v),i,j,\theta_i}$ denoting all possible disturbance effects on system (6) during vth phase, where $t(v)=k_v^1+T^{(v)}$, θ_i is the length of dwell-time portion of vth phase and i denotes the subsystem during the dwell-time. Let $\mathcal{A}_{t_1}^{t_2}:=\prod_{l=t_1}^{t_2}\bar{A}_{\sigma(l),\hat{\sigma}(l)}$, $t_1\le t_2$, t_1 and $t_2\in\mathbb{Z}_+$, $\{\sigma(l)\,|\,l\in\mathbb{Z}_{[t_1,t_2]}\}\in\zeta_{[\tau]^T}(t_2)$ and $\{\hat{\sigma}(l)\,|l\in\mathbb{Z}_{[t_1,t_2]}\}\in\zeta_{[\tau,\mathcal{T}]^T}(t_2)$,

$$\begin{aligned}S_{t(v),i,j,\theta_i}=&\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}\bar{A}_{i,j}^{\mathcal{T}_i-1}B_i\mathbb{W}\\&\oplus\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}\bar{A}_{i,j}^{\mathcal{T}_i-2}B_i\mathbb{W}\oplus\cdots\\&\oplus\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}B_i\mathbb{W}\\&\oplus\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i-1}B_i\mathbb{W}\oplus\cdots\\&\oplus\mathcal{A}_{k_v^1}^{t(v)}B_i\mathbb{W}\\&\oplus\mathcal{A}_{k_v^1+1}^{t(v)}B_{\sigma(l-1)}\mathbb{W}\oplus\cdots\\&\oplus\mathcal{A}_{t(v)}^{t(v)}B_{\sigma(t(v)-1)}\mathbb{W}\\&\oplus B_{\sigma(t(v))}\mathbb{W}.\end{aligned}$$

Define a set $\Theta_i=\{\tau_i,\tau_i+1,\ldots,2\tau_i-1\}$ and consider

$$\begin{aligned}O_{v+1}=co\Big\{&\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}\bar{A}_{i,j}^{\mathcal{T}_i}O_v\oplus S_{t(v),i,j,\theta_i}:\\&(i,j)\in\mathcal{I}\times\mathcal{I},\ T^{(v)}\in\mathbb{Z}_{[0,T]},\ \theta_i\in\Theta_i,\\&\{\sigma(l)\,|\,l\in\mathbb{Z}_{[k_v^1,t(v)]}\}\in\zeta_{[\tau]^T}(t(v))\Big\}\end{aligned}\tag{37}$$

where $(\sigma(k_v),\sigma(k_v-1))=(i,j)$. Let $O_0=\Lambda$ or $\{0\}$, where $\Lambda:=co\{\mathcal{A}_2^kB_{\sigma(1)}\mathbb{W}\oplus\cdots\oplus\bar{A}_{\sigma(k),\hat{\sigma}(k)}B_{\sigma(k-1)}\mathbb{W}\oplus B_{\sigma(k)}\mathbb{W}:k\in\mathbb{Z}_{[1,T]}\}$ If letting $O_0=\Lambda$, the case of a persistence portion existing before the first phase is considered and $O_0=\{0\}$ denotes the contrary. Set $\mathcal{S}:=co\{S_{k_v^1+T^{(v)},i,j,\theta_i}:T^{(v)}\in\mathbb{Z}_{[0,T]},(i,j)\in\mathcal{I}\times\mathcal{I},\theta_i\in\Theta_i\}$, then

$$\begin{aligned}O_{v+1}\subseteq co\Big\{&\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}\bar{A}_{i,j}^{\mathcal{T}_i}O_v\oplus\mathcal{S}:\\&(i,j)\in\mathcal{I}\times\mathcal{I},T^{(v)}\in\mathbb{Z}_{[0,T]},\theta_i\in\Theta_i\\&\{\sigma(l)\,|\,l\in\mathbb{Z}_{[k_v^1,t(v)]}\}\in\zeta_{[\tau]^T}(t(v))\Big\}.\end{aligned}\tag{38}$$

Iterating (38) from v to 0, $v\in\mathbb{Z}_{\ge1}$, one gets

$$\begin{aligned}\Gamma_v=co\Big\{&[\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{h,h}^{\theta_h-\mathcal{T}_h}\bar{A}_{h,\hat{\sigma}(k_v)}^{\mathcal{T}_h}]\\&\times[\mathcal{A}_{k_{v-1}^1}^{t(v-1)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}\bar{A}_{i,\hat{\sigma}(k_{v-1})}^{\mathcal{T}_i}]\times\cdots\\&\times[\mathcal{A}_{k_2^1}^{t(2)}\bar{A}_{j,j}^{\theta_j-\mathcal{T}_j}\bar{A}_{j,\hat{\sigma}(k_2)}^{\mathcal{T}_j}]\\&\times[\mathcal{A}_{k_1^1}^{t(1)}\bar{A}_{m,m}^{\theta_m-\mathcal{T}_m}\bar{A}_{m,\hat{\sigma}(k_1)}^{\mathcal{T}_m}]O_0\\&\oplus[\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{h,h}^{\theta_h-\mathcal{T}_h}\bar{A}_{h,\hat{\sigma}(k_v)}^{\mathcal{T}_h}]\\&\times[\mathcal{A}_{k_{v-1}^1}^{t(v-1)}\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}\bar{A}_{i,\hat{\sigma}(k_{v-1})}^{\mathcal{T}_i}]\times\cdots\\&\times[\mathcal{A}_{k_2^1}^{t(2)}\bar{A}_{j,j}^{\theta_j-\mathcal{T}_j}\bar{A}_{j,\hat{\sigma}(k_2)}^{\mathcal{T}_j}]\mathcal{S}\oplus\cdots\\&\oplus[\mathcal{A}_{k_v^1}^{t(v)}\bar{A}_{h,h}^{\theta_h-\mathcal{T}_h}\bar{A}_{h,\hat{\sigma}(k_v)}^{\mathcal{T}_h}]\mathcal{S}\oplus\mathcal{S}\Big\}\end{aligned}$$

and $O_v\subseteq\Gamma_v$. Given the fact that system (5) is GUAS with asynchronous MPDT set $[\tau,\mathcal{T}]^T$, it follows that $\hat{x}_{k_{v+1}}=\widehat{A}\hat{x}_{k_v}$ is asymptotically stable under any admissible switching sequence in $\zeta_{[\tau,\mathcal{T}]^T}(k_{v+1})$, where $\widehat{A}\in\Xi(\Theta_i,\mathcal{T}_i,T):=\{(\prod_{l=1}^t\bar{A}_{\sigma(l),\hat{\sigma}(l)})\bar{A}_{i,i}^{\theta_i-\mathcal{T}_i}\bar{A}_{i,j}^{\mathcal{T}_i}|t\in\mathbb{Z}_{[0,T]},\ \theta_i\in\Theta_i,\ (i,j)\in\mathcal{I}\times\mathcal{I}\}$. With a similar use of the finite set Θ_i in [23], the evolution of subsystems from k_v to k_{v+r}, $r\in\mathbb{Z}_{\ge1}$, under any switching sequence in $\zeta_{[\tau,\mathcal{T}]^T}(k_{v+r})$ can be represented by the sequential product of matrices in $\Xi(\Theta_i,\mathcal{T}_i,T)$. Consequently, there exist constants $\epsilon\in(0,1)$ and $\eta>0$ satisfying $\mathcal{S}\subseteq\eta\mathcal{B}^n$, such that $\widehat{A}\mathcal{S}\subseteq\eta\epsilon\mathcal{B}^n$. Considering $O_0=\Lambda$ or $\{0\}$, $\Lambda\subseteq\mathcal{S}$, it yields that

$$O_v\subseteq\Gamma_v\subseteq\eta(\epsilon^v+\cdots+\epsilon+1)\mathcal{B}^n.\tag{39}$$

Therefore, by (38) and (39), $O_v\subseteq O_{v+1}$ and that O_v is bounded by $(\eta/(1-\epsilon))\mathcal{B}^n$ as $v\to\infty$ are guaranteed, respectively. Thus there exists a $[\tau,\mathcal{T}]^T$ dependent limit set O_∞ for the set sequence $\{O_v|v\in\mathbb{Z}_+\}$. In consequence, for system (6), for any $e_0\in O([\tau,\mathcal{T}]^T)=O_\infty$, $e_k\in O([\tau,\mathcal{T}]^T)$ for any admissible asynchronous MPDT switching $\zeta_{[\tau,\mathcal{T}]^T}$ with $w(k)\in\mathbb{W}$, $k\in\mathbb{Z}_{\ge1}$. ■

By the use of non-QTD observer gains, the evolution of the system (6) under all admissible switching sequences can be represented by combinations of matrices in $\Xi(\Theta_i,\mathcal{T}_i,T)$ with a finite set Θ_i, while such comparable sets will not exist in the QTD case [21].

To compute the asynchronous MPDT RPI set $O([\tau,\mathcal{T}]^T)$, one step reachable set of $\mathcal{X}$ in subsystem Ω_i with observer L_j is defined first, $\mathcal{R}_1^{i,j}(\mathcal{X},\mathbb{W}):=\{\bar{A}_{i,j}x\oplus B_iw|x\in\mathcal{X},\ w\in\mathbb{W}\}=\bar{A}_{i,j}\mathcal{X}\oplus B_i\mathbb{W}$, $\forall(i,j)\in\mathcal{I}\times\mathcal{I}$, and an N-step reachable set is accordingly defined as $\mathcal{R}_N^{i,j}(\mathcal{X},\mathbb{W}):=\mathcal{R}_1^{i,j}(\mathcal{R}_{N-1}^{i,j}(\mathcal{X},\mathbb{W}),\mathbb{W})$, where $\mathcal{R}_0^{i,j}(\mathcal{X},\mathbb{W})=\mathcal{X}$, $N\in\mathbb{Z}_{\ge1}$. The expanded form is $\mathcal{R}_N^{i,j}(\mathcal{X},\mathbb{W})=\bar{A}_{i,j}^N\mathcal{X}\oplus\bar{A}_{i,j}^{N-1}B_i\mathbb{W}\oplus\cdots\oplus B_i\mathbb{W}$. In order to compute the reachable set under all admissible asynchronous MPDT switching sequences, three reachable-set operators $\widehat{\mathcal{R}}_i(\cdot,\mathbb{W})$, $\overline{\mathcal{R}}(\cdot,\mathbb{W})$ and $\widetilde{\mathcal{R}}(\cdot,\mathbb{W})$ are defined for the periods of asynchronous delay, dwell-time portion and persistence portion respectively as follows:

$$\widehat{\mathcal{R}}_i(\cdot,\mathbb{W}) = \bigcup_{j\in\mathcal{I}} \mathcal{R}^{i,j}_{\mathcal{T}_i}(\cdot,\mathbb{W})$$

$$\overline{\mathcal{R}}(\cdot,\mathbb{W}) = \bigcup_{i\in\mathcal{I}}\bigcup_{\theta_i\in\Theta_i} \mathcal{R}^{i,i}_{\theta_i-\mathcal{T}_i}\left(\widehat{\mathcal{R}}_i(\cdot,\mathbb{W}),\mathbb{W}\right)$$

$$\widetilde{\mathcal{R}}(\cdot,\mathbb{W}) = \bigcup_{t\in\mathbb{Z}_{[0,T]}}\ \bigcup_{\{\sigma(k)\}\in\zeta_{[\tau,\mathcal{T}]^T}} \mathcal{R}_1^{\sigma(t),\hat{\sigma}(t)}(\mathcal{R}_1^{\sigma(t-1),\hat{\sigma}(t-1)} \times(\cdots\mathcal{R}_1^{\sigma(0),\hat{\sigma}(0)}(\cdot,\mathbb{W}),\ldots,\mathbb{W}),\mathbb{W})$$

where $k\in\mathbb{Z}_{[0,t]}$. Based on Theorem 2, the algorithm for computation of the asynchronous MPDT RPI set $O([\tau,\mathcal{T}]^T)$ is proposed as follows:

Algorithm 1 Computation of $O([\tau,\mathcal{T}]^T)$

Input: $\mathbb{W}$; $\{\overline{A}_{i,j}\}$; $[\tau,\mathcal{T}]^T$
1: initial $v=0$, $O_v = co\{\widetilde{\mathcal{R}}(\{0\},\mathbb{W})\}$ or $\{0\}$;
2: **repeat**
3: $\quad O_{v+1} = co\{\widetilde{\mathcal{R}}(\overline{\mathcal{R}}(O_v,\mathbb{W}),\mathbb{W})\}$;
4: $\quad v = v+1$;
5: **until** $O_{v+1}\equiv O_v$
Output: $O([\tau,\mathcal{T}]^T) = O_{v+1}$

It should be noted that the convergence of Algorithm 1 is guaranteed by the existence of $O([\tau,\mathcal{T}]^T)$. By Definition 4, if $e_0\in O([\tau,\mathcal{T}]^T)$, e_k only has to stay inside $O([\tau,\mathcal{T}]^T)$ at switching instants $\{k_s|s\in\mathbb{Z}_+\}$, so it is allowable that the states pass in and out $O([\tau,\mathcal{T}]^T)$ multiple times during each phase. Consider the reachable sets in the asynchronous MPDT switching case, during dwell-time portions, the states are driven into $O([\tau,\mathcal{T}]^T)$ owing to the effect of matched observers, while in persistence portions and asynchronous observation, frequent switching and unmatched observers generally take an opposite effect. It yields that $\widehat{\mathcal{R}}_i(O([\tau,\mathcal{T}]^T),\mathbb{W})$ and $\mathcal{R}^{i,i}_t(\widehat{\mathcal{R}}_i(O([\tau,\mathcal{T}]^T),\mathbb{W}),\mathbb{W})$ are not necessarily the subsets of $O([\tau,\mathcal{T}]^T)$, $t\in\mathbb{Z}_{[0,\tau_i-\mathcal{T}_i]}$, $\forall i\in\mathcal{I}$.

Thus, let

$$\mathcal{G}_i\left([\tau,\mathcal{T}]^T\right) = co\Bigg\{\bigcup_{t\in\mathbb{Z}_{[0,\tau_i-\mathcal{T}_i]}}(\mathcal{R}^{i,i}_t(\widehat{\mathcal{R}}_i(O([\tau,\mathcal{T}]^T),\mathbb{W}),\mathbb{W})) \times\bigcup_{j\in\mathcal{I}}\bigcup_{t\in\mathbb{Z}_{[0,\mathcal{T}_i]}}\mathcal{R}^{i,j}_t(O([\tau,\mathcal{T}]^T),\mathbb{W})\Bigg\}.$$

If $e_{k_s}\in O([\tau,\mathcal{T}]^T)\subseteq\mathcal{G}_i([\tau,\mathcal{T}]^T)$, we have $e_k\in\mathcal{G}_i([\tau,\mathcal{T}]^T)$, $k\in\mathbb{Z}_{[k_s,k_{s+1}]}$, $i=\sigma(k_s)$. Moreover, the asynchronous MPDT GRPI set is obtained as $\mathcal{G}([\tau,\mathcal{T}]^T) := \bigcup_{i\in\mathcal{I}}\mathcal{G}_i([\tau,\mathcal{T}]^T)$, based on which the stability criterion of (6) can be established in the sense of Definition 7 in the following theorem.

Theorem 3: Consider the observer error system (4). Suppose that a set of non-QTD observers exist for the nominal system (4) with asynchronous MPDT switching $\zeta_{[\tau,\mathcal{T}]^T}$. Then the set $\mathcal{G}([\tau,\mathcal{T}]^T)$ is GUAS for the system (4).

Proof: If there exist a set of observers such that system (5) is GUAS with the $\zeta_{[\tau,\mathcal{T}]^T}$ satisfying (22), then it follows from Definition 6 that $\|z_k\|\le\kappa(\|z_0\|)$, $\forall k\in\mathbb{Z}_{\ge0}$ and $\|z_k\|\to0$ as $k\to\infty$, where $\kappa\in\mathcal{K}_\infty$. With $\varepsilon_k = z_k+e_k$ and $e_k\in\mathcal{G}([\tau,\mathcal{T}]^T)$, it follows that $\|\varepsilon_k\|_{\mathcal{G}([\tau,\mathcal{T}]^T)} = d(z_k+e_k,\mathcal{G}([\tau,\mathcal{T}]^T))\le d(z_k+e_k, e_k) = \|z_k\|\le\kappa(\|z_0\|)$ and $\|\varepsilon_k\|_{\mathcal{G}([\tau,\mathcal{T}]^T)}\to0$ as $k\to\infty$. For any z_0 and ε_0, there exists a positive constant α such that $\alpha\le\|\varepsilon_0\|_{\mathcal{G}([\tau,\mathcal{T}]^T)}/\|z_0\|$, thus we have $\kappa(\|z_0\|)\le\kappa(\|\varepsilon_0\|_{\mathcal{G}([\tau,\mathcal{T}]^T)}/\alpha)$ and $\|\varepsilon_k\|_{\mathcal{G}([\tau,\mathcal{T}]^T)}\le\kappa(\|\varepsilon_0\|_{\mathcal{G}([\tau,\mathcal{T}]^T)}/\alpha)$, $\kappa(\cdot/\alpha)\in\mathcal{K}_\infty$, which indicates that $\mathcal{G}([\tau,\mathcal{T}]^T)$ is GUAS for the system (4) in the sense of Definition 7. ■

Remark 5: For the defined and calculated asynchronous MPDT GRPI set $\mathcal{G}([\tau,\mathcal{T}]^T)$, once the state trajectory of system (4) enters $\mathcal{G}([\tau,\mathcal{T}]^T)$, it will always remain inside. Accordingly, let the asynchronous MPDT GRPI set $\mathcal{G}([\tau,\mathcal{T}]^T)$ be the cross section of a uniform tube, of which the center is the state of the nominal observer error system (5), then all the trajectories of the disturbed observer error system (4) will be contained in the uniform tube.

Remark 6: When the union operations related to the asynchronous switching in the calculation of asynchronous MPDT RPI set $O([\tau,\mathcal{T}]^T)$ and GRPI set $\mathcal{G}([\tau,\mathcal{T}]^T)$ are removed, the sets in the synchronous switching case will be obtained. Thus the MPDT RPI and GRPI sets in the case of synchronous switching can be regarded as particular cases of the asynchronous ones and the asynchronous sets are more general than the synchronous ones.

To show the effectiveness of Theorem 2 and Theorem 3, a numerical example is presented here.

Example 2: Consider a switched linear system with two subsystems

$$A_1 = \begin{bmatrix} 0.984 & 0.120 \\ -0.072 & 0.924 \end{bmatrix},\quad A_2 = \begin{bmatrix} -0.784 & 0.441 \\ 0.0784 & -0.666 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad B_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},\quad C_2 = \begin{bmatrix} 0.3 & 0.4 \end{bmatrix}. \tag{40}$$

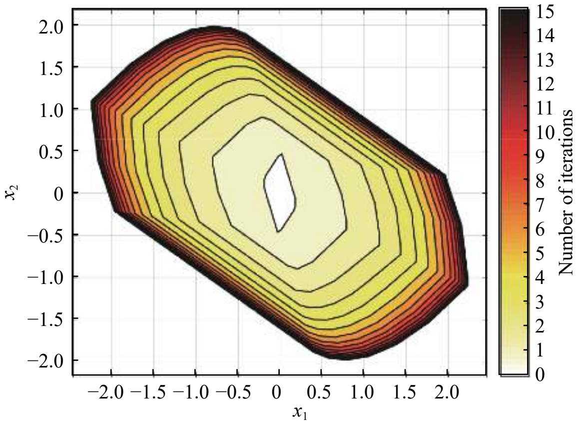
The aim is to design a non-QTD observer and calculate the corresponding asynchronous RPI and GRPI sets for the system (40). With the searched constants $\alpha = 0.3025$, $\beta = 1.2506$ and $\mu = 1.2502$, the non-QTD observer gains are obtained as

$$L_1 = [0.8696; 0.7599],\quad L_2 = [0.0585; -0.6399].$$

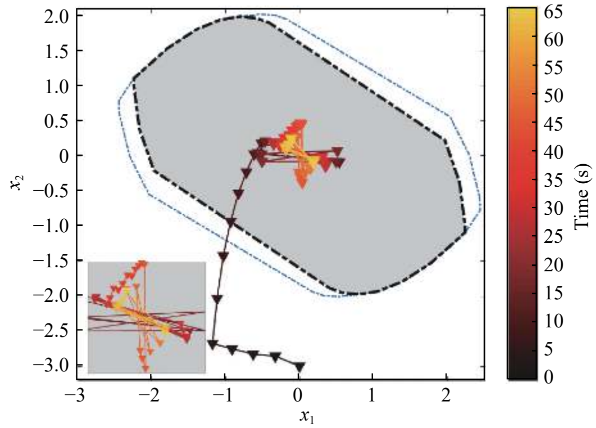
Let the disturbance input be restricted as $\|\mathbb{W}\|_\infty\le0.1$, the asynchronous MPDT RPI set O_v and GRPI set $\mathcal{G}([\tau,\mathcal{T}]^T)$ are derived and illustrated in Fig. 4. We suppose that there exists a persistence portion before the first phase starts, causing O_0 to be non-zero, which is shown in Fig. 4 (a). It is shown that O_v almost converges after 10 iterations and eventually reaches convergence at the 14th iteration. The state trajectory of the difference system in (40) is illustrated in Fig. 4 (b). It is observed that once the trajectory enters $\mathcal{G}([\tau,\mathcal{T}]^T)$, the states always remain in $O([\tau,\mathcal{T}]^T)$ at switching instants and stay inside the GRPI $\mathcal{G}([\tau,\mathcal{T}]^T)$ all the time, though the trajectory shows some level of chattering when asynchronous delays or fast switching occurs.

IV. Experiment With the Space Robot Manipulator

In this section, our method is tested on a space robot manipulator (SRM) model to demonstrate the validity and applicability. Let us consider an SRM model with the



(a)



(b)

Fig. 4. Visualization of MPDT GRPI set and evolution of the difference system. (a) shows the computation process of Algorithm 2, where the white set denotes the initial set O_0 and the darkest set denotes $O([\tau,\mathcal{T}]^T)$. In (b), evolution of state trajectory is demonstrated in 2D view, where the inner shadowed set is $O([\tau,\mathcal{T}]^T)$ and the outer blue dashed line denotes the $\mathcal{G}([\tau,\mathcal{T}]^T)$. A zoom-in display is presented in the bottom of (b) to have a detailed view of the state trajectory.

following rigid-body dynamics equations as proposed in [31]

$$\begin{cases} \mathcal{N}^2 J_{\text{in}}\ddot{\Omega} + J_{\text{out}}(\ddot{\Omega}+\ddot{\delta}) + \beta(\dot{\Omega}+\dot{\delta}) = T_{\text{eff}} \\ J_{\text{out}}(\ddot{\Omega}+\ddot{\delta}) + \beta(\dot{\Omega}+\dot{\delta}) = T_{\text{def}} \end{cases} \tag{41}$$

where Ω, δ, T_{eff} and T_{def} are variables representing the joint angle of inertial axis, the joint angle of the output axis, the effective joint input torque and deformation torque of the gearbox, respectively; $\mathcal{N}$ is the gearbox ratio and β is the damping coefficient; J_{in} and J_{out} stand for the inertia of the input axis and output system, respectively. As in [31], the actuator model of the motor plus the gearbox is formulated as follows:

$$T_{\text{eff}} = \mathcal{N} K_t i_c, \quad T_{\text{def}} = c\delta \tag{42}$$

where K_t denotes the motor torque constant and c denotes the spring constant; variable i_c stands for the motor current, which is also the control input u to the system. We assume no drivetrain loss in the model.

Let $x = [\Omega, \dot{\Omega}, \delta, \dot{\delta}]'$ be the state of the SRM system (41) and $y = [\Omega+\delta, \mathcal{N}\dot{\Omega}]$ be the output. Suppose there exists an additive disturbance w in the control input channel, then the state-space model of the system can be described as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{c}{\mathcal{N}^2 J_{\text{in}}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{\beta}{J_{\text{out}}} & -\left(\dfrac{c}{\mathcal{N}^2 J_{\text{in}}} + \dfrac{c}{J_{\text{out}}}\right) & -\dfrac{\beta}{J_{\text{out}}} \end{bmatrix} x + \begin{bmatrix} 0 \\ \dfrac{K_{\text{t}}}{\mathcal{N} J_{\text{in}}} \\ 0 \\ -\dfrac{K_{\text{t}}}{\mathcal{N} J_{\text{in}}} \end{bmatrix} u + \begin{bmatrix} 0 \\ \dfrac{K_{\text{t}}}{\mathcal{N} J_{\text{in}}} \\ 0 \\ -\dfrac{K_{\text{t}}}{\mathcal{N} J_{\text{in}}} \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \mathcal{N} & 0 & 0 \end{bmatrix} x. \tag{43}$$

It is known that the motor torque constant K_t and inertia of the input axis J_{in} may change abruptly when the SRM system encounters failures. Then (43) can be modeled as a switched system under persistent dwell-time switching. In this paper, we consider generalizing the allowable switching to a relatively broader scope by MPDT switching. Suppose there are two modes in the switched system, and the corresponding parameter values are

$$K_t^{\{1,2\}} = \{0.6, 0.13\}$$
$$J_{\text{in}}^{\{1,2\}} = \{1.1, 0.8\} \times 10^{-3}.$$

To apply our method to the continuous model, the system (43) is discretized with a sampling period $T_{\text{s}} = 0.1$(s) and the asynchronous MPDT switching is constrained as $\tau_{\min} = 4$, $\mathcal{T}_1 = \mathcal{T}_2 = 2$ and $T^{(s)} \le 4$. The non-QTD asynchronous observers are designed as in Corollary 2. By setting $\mathcal{T}_i = 0$, $\forall i \in \mathcal{I}$ in Corollary 1, non-QTD synchronous observers are obtained and applied as well for comparison. With disturbances restricted as $\|\mathbb{W}\|_\infty \le 0.01$, the asynchronous RPI set $O([\tau,\mathcal{T}]^T)$ and GRPI set $\mathcal{G}([\tau,\mathcal{T}]^T)$ are obtained and shown in Fig. 5. Given the fact that the SRM model has four dimensions, resulting in that $O([\tau,\mathcal{T}]^T)$ and $\mathcal{G}([\tau,\mathcal{T}]^T)$ can not be visualized directly, the figures shown here are the projection of $O([\tau,\mathcal{T}]^T)$ and $\mathcal{G}([\tau,\mathcal{T}]^T)$ in two orthogonal planes.

The discretized system is unstable without control input and performance of the observer is difficult to determine in this case. Hence, an observer-based non-switching state feedback controller $u_k = K\hat{x}_k$ is designed by solving a constrained feasibility problem in the form of LMIs, and implemented as in Fig. 2. The controller gain is obtained as K = [6.7876, 13.9042, 448.0034, 22.9035]. The constants searched by bisection are also listed here: for the observer designed by Corollary 2, $\alpha = 0.9612$, $\beta = 1.01$ and $\mu = 1.0096$; for that of Corollary 1, $\alpha = 0.05$ and $\mu = 1.25$.

The evolution of state trajectories of the disturbed and nominal observer error system with the asynchronous and synchronous observers is shown in Fig. 6, where the four

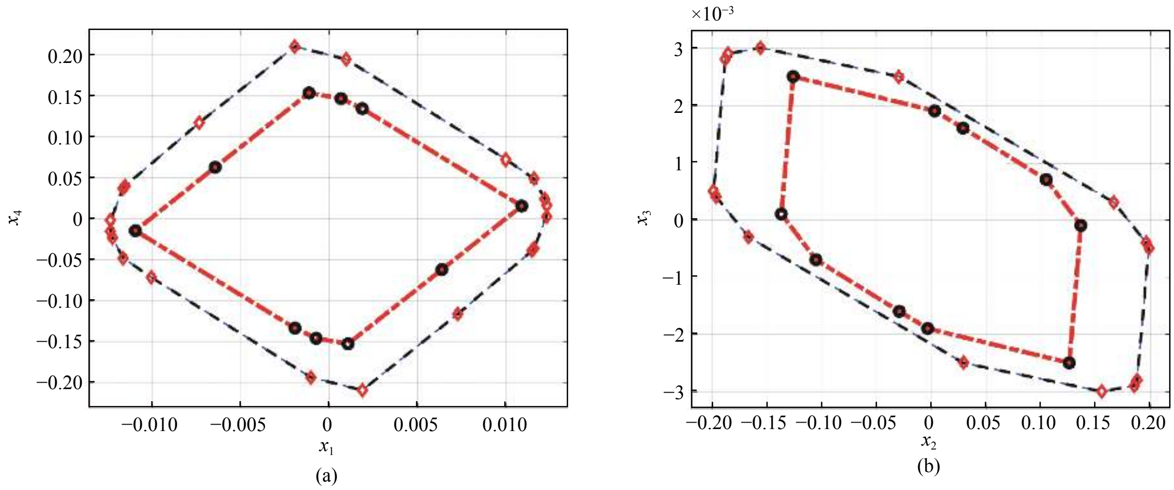


Fig. 5. Visualization of the obtained asynchronous RPI set $\mathcal{O}([\tau,\mathcal{T}]^T)$ (the red dashed line) and GRPI set $\mathcal{G}([\tau,\mathcal{T}]^T)$ (the blue dashed line) of the SRM system. (a) Projection of $\mathcal{O}([\tau,\mathcal{T}]^T)$ and $\mathcal{G}([\tau,\mathcal{T}]^T)$ in the plane formed by x_1 and x_4; (b) Projection of $\mathcal{O}([\tau,\mathcal{T}]^T)$ and $\mathcal{G}([\tau,\mathcal{T}]^T)$ in the plane formed by x_2 and x_3.

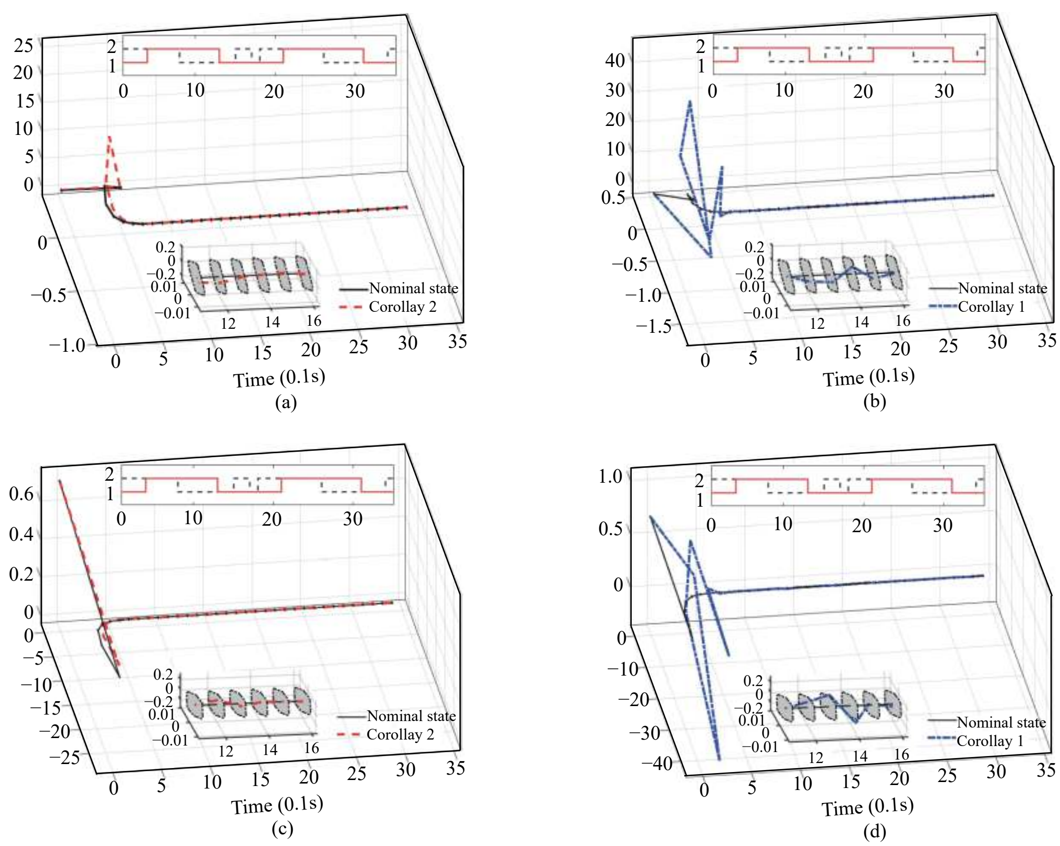


Fig. 6. Evolution of state trajectories of the observer error system of the SRM. The nominal trajectories are drawn in black, while the red and blue dashed lines stand for the trajectories of disturbed observer error system resulting from Corollary 2 and Corollary 1, respectively. To show the four dimensional trajectories, we project them in two orthogonal directions correspondingly with Fig. 5, and time is added as the third dimension generating the three dimensional trajectories. (a) and (b) show the state trajectories in the space formed by x_1, x_4 and time; (c) and (d) show the state trajectories in the space formed by x_2, x_3 and time.

dimensional trajectories are projected in the same manner as in Fig. 5. The convergence in final phase can be found in the zoom-in display in Fig. 6. In the presence of asynchronous delays and disturbances, it is noted that there are larger state overshoots in the synchronous observer case than in the asynchronous one. Additionally, as shown in the zoom-in display, the asynchronous observers are capable of attracting the observer errors into $\mathcal{G}([\tau,\mathcal{T}]^T)$ and maintaining it consistently, while the synchronous ones fail.

Remark 7: The experiment and examples above show that our approach is effective for the observation of switched systems in presence of asynchronous switchings and l_∞

disturbances. However, it should be noted that when applied to systems with a large number of subsystems, the increased number of constraints may inherently affect feasibility of the LMIs and application of the proposed method.

V. Conclusions

The asynchronous observer design problem of a class of discrete-time switched linear systems with additive l_∞ disturbances under MPDT switching is investigated. The existence condition of asynchronous QTD observers for the nominal observer error system to be GUAS is proposed. A numerical example is presented to demonstrate the effectiveness of asynchronous observers compared to the synchronous ones. In the presence of l_∞ disturbances, the asynchronous MPDT RPI set and the corresponding algorithm are presented, ensuring that the state of the difference system remain inside the RPI set at switching instants. Furthermore, the asynchronous MPDT GRPI set is determined as the cross section of a uniform tube of the observer error system, of which the asymptotic stability is demonstrated in the sense of converging to the asynchronous MPDT GRPI set. Eventually, an SRM example is introduced to validate the effectiveness of the proposed method.

Appendix A
Relation Graph of Theorems and Corollaries

The relationship among the proposed theorems and corollaries is given in Fig. 7. Lemma 1 is the basis for all other later derivations and provides a sufficient criterion for the GUAS of switched systems under general asynchronous MPDT switching. In Theorem 1, considering a class of quadratic QTD Lyapunov functions, the design of the asynchronous QTD observer is proposed guaranteeing the nominal observer error system to be GUAS. Let the asynchronous delays in Theorem 1 be zero, the synchronous observer design is obtained in Corollary 1. Based on Theorem 1, Corollary 2 gives the non-QTD asynchronous observer design. In Theorem 2, the asynchronous MPDT RPI set is determined by applying the results developed in Corollary 2. Based on the proposed asynchronous MPDT RPI set, the asynchronous GRPI set is further developed. Theorem 3 proves that the asynchronous MPDT GRPI set is GUAS for the observer error system in the sense of Definition 7.

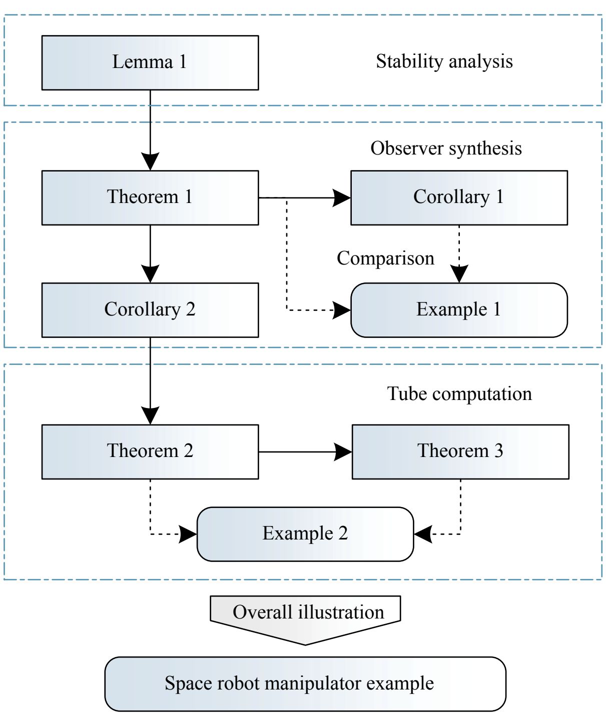


Fig. 7. Relation among the obtained criteria.

Appendix B
Proof of Lemma 1

Proof: First of all, if $\beta < 1$, then the system falls into the class of switched systems under synchronous switching (where the energy always decays during each subsystem), and this lemma transforms to a stability criterion for the corresponding system to be GUAS [17]. Thus the proof boils down to the case $\beta \geq 1$.

Consider $\sigma(k_s) = i$, $\sigma(k_s^1 + T^{(s)} - 1) = l$, and $\sigma(k_s^1 + T^{(s)}) = j$ in the s^{th} phase of the MPDT switching signal. Suppose an arbitrary switching occurs within $T^{(s)}$, it follows from (7)–(12) that

$$\begin{aligned} &V_j(x_{k_s^1+T^{(s)}},0)\\ &\quad\leq \mu V_l(x_{k_s^1+T^{(s)}},T_l)\\ &\quad\leq \mu\alpha^{T_l-\mathcal{T}_l}V_l(x_{k_s^1+T^{(s)}-T_l+\mathcal{T}_l},\mathcal{T}_l)\\ &\quad\leq \mu\alpha^{T_l-\mathcal{T}_l}\beta_l^{\mathcal{T}_l}V_l(x_{k_s^1+T^{(s)}-T_l},0). \end{aligned}$$

Then, iterating from k_{s+1} to k_s^1 with the following procedure, one gets

$$\begin{aligned} &V_j(x_{k_s^1+T^{(s)}},0)\\ &\quad\leq \prod_{r=1}^{Q(k_s,k_{s+1})}(\mu\alpha^{T_{\sigma_r}-\mathcal{T}_{\sigma_r}}\beta^{\mathcal{T}_{\sigma_r}})V_{\sigma(k_s^1)}(x_{k_s^1},0)\\ &\quad\leq \prod_{r=1}^{Q(k_s,k_{s+1})}(\mu\alpha^{T_{\sigma_r}-\mathcal{T}_{\sigma_r}}\beta^{\mathcal{T}_{\sigma_r}})\mu V_i(x_{k_s^1},\tau_i)\\ &\quad\leq \prod_{r=1}^{Q(k_s,k_{s+1})}(\mu\alpha^{T_{\sigma_r}-\mathcal{T}_{\sigma_r}}\beta^{\mathcal{T}_{\sigma_r}})\mu\alpha^{k_s^1-k_s+\mathcal{T}_i}\beta^{\mathcal{T}_i}V_i(x_{k_s},0)\\ &\quad\leq \prod_{r=1}^{Q(k_s,k_{s+1})}(\mu\alpha^{T_{\sigma_r}-\mathcal{T}_{\sigma_r}}\beta^{\mathcal{T}_{\sigma(k_s^r)}})\mu\alpha^{\tau_i+\mathcal{T}_i}\beta^{\mathcal{T}_i}V_i(x_{k_s},0) \end{aligned}$$

where $\sigma_r = \sigma(k_s^r)$ and $Q(k_s,k_{s+1})$ denotes the number of switchings between k_s and k_{s+1}.

Since $\beta \geq 1 > \alpha$, it holds that $\alpha^{T_i-\mathcal{T}_i}\beta^{\mathcal{T}_i} < \beta^{T_i}$, $\forall\sigma(k) = i \in \mathcal{I}$. We have $V_j(x_{k_s^1+T^{(s)}},0) \leq \mu^{T^{(s)}+1}\beta^{T^{(s)}}\alpha^{\tau_i-\mathcal{T}_i}\beta^{\mathcal{T}_i}V_i(x_{k_s},0) < \mu^{T+1}\times \beta^T\alpha^{\tau_i-\mathcal{T}_i}\beta^{\mathcal{T}_i}V_i(x_{k_s},0)$. Let $\lambda_{\max} := \max_{i\in\mathcal{I}}\lambda_i$, combined with (13), one has $\lambda_{\max} \leq 1$. Take the fact that a period of persistence may exist before the first phase into account, it follows $V_{\sigma(k_{s+1})}(x_{k_{s+1}},0) \leq \lambda_{\max}V_{\sigma(k_s)}(x_{k_s},0) \leq \cdots \leq \lambda_{\max}^{s-1}\mu^T\beta^T\times V_{\sigma(0)}(x_0,0)$. From (7), $\|x_k\| \leq \kappa_1^{-1}(\lambda_{\max}^{s-1}\mu^T\beta^T\kappa_2(\|x_k\|))$ holds. Thus, with (7)–(12), $\|x_k\| \leq \kappa_3(\|x_k\|)$, where $\kappa_3(\|\cdot\|) := \kappa_1^{-1}(\lambda_{\max}^{s-1}\mu^T\beta^T\kappa_2(\|\cdot\|))$. Therefore the global uniform asymptotic stability of the switched system $x_{k+1} = f_{\sigma(k)}(x_k)$ can be inferred by Definition 6. ■

APPENDIX C
PROOF OF THEOREM 1

Proof: First of all, for matrix $P_i(\varphi+1)\in\mathbb{S}^{n_x}_{>0}$, $\varphi\in\mathbb{Z}_{[0,\tau_i]}$, $\forall i\in\mathcal{I}$, from the fact that $[(P_i-P'_j)P_i^{-1}(P_i-P_j)]\geq 0$, we have $(P_i(\theta_1)-2P_j(\theta_3))\geq -P'_j(\theta_3)P_i^{-1}(\theta_1)P_j(\theta_3)$. Thus the following inequalities hold:

$$\hat{\Phi}_{i,j}(\theta_1,\theta_2,\theta_3)\leq\Phi_{i,j}(\theta_1,\theta_2,\theta_3) \tag{44}$$

$$\hat{\Upsilon}_i(\theta_1,\theta_2)\leq\Upsilon_i(\theta_1,\theta_2) \tag{45}$$

where

$$\hat{\Phi}_{i,j}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} -P'_j(\theta_3)P_i^{-1}(\theta_1)P_j(\theta_3) & P_j(\theta_3)\big(A_i-L_j(\theta_3)C_i\big) \\ * & -\beta P_i(\theta_2)\end{bmatrix}$$

$$\hat{\Upsilon}_i(\theta_1,\theta_2) = \begin{bmatrix} -P'_i(\theta_2)P_i^{-1}(\theta_1)P_i(\theta_2) & P_i(\theta_2)(A_i-L_i(\theta_2)C_i) \\ * & -\alpha P_i(\theta_2)\end{bmatrix}.$$

With (14), (15), (44) and (45), the following inequalities are guaranteed:

$$\hat{\Phi}_{i,j}(\varphi+1,\varphi,k)\leq 0 \tag{46}$$

$$\hat{\Upsilon}_i(\varphi+1,\varphi)\leq 0. \tag{47}$$

Perform congruence transformation to (16), (46) and (47) with $\mathrm{diag}\{P_i^{-1}(\tau_i),I\}$, $\mathrm{diag}\{P_j^{-1}(k),I\}$ and $\mathrm{diag}\{P_i^{-1}(\varphi),I\}$, respectively. With the Schur complement, one gets (8)–(12) for $V_i(e_k,\varphi_k)=e_k^TP_i(\varphi)e_k$, $\varphi=\varphi_k\in\mathbb{Z}_{[0,\tau_i]}$, $\forall i\in\mathcal{I}$.

With (22) and the fact that $0<\alpha<1$, $\beta\geq 1$, $\alpha^{\tau_i-\mathcal{T}_i}\leq\alpha^{\tau_{\min}-\mathcal{T}_{\max}}$ and $\beta^{\mathcal{T}_i}\leq\beta^{\mathcal{T}_{\max}}$, it follows that $\mu^{T+1}\beta^T\alpha^{\tau_i-\mathcal{T}_i}\beta^{\mathcal{T}_i}\leq\vartheta\leq 1$, $\forall i\in\mathcal{I}$ and (13) is guaranteed. Then by Lemma 1, the nominal observer error system (5) is GUAS for asynchronous MPDT switching signals satisfying (22). ■

REFERENCES

[1] L. Ni and D. W. Wang, "A gain-switching control scheme for positionerror-based bilateral teleoperation: Contact stability analysis and controller design," *The Int. J. Robotics Research*, vol. 23, no. 3, pp. 255–274, 2004.

[2] K. S. Narendra, J. Balakrishnan, and M. K. Ciliz, "Adaptation and learning using multiple models, switching, and tuning," *IEEE Control Systems*, vol. 15, no. 3, pp. 37–51, 1995.

[3] Y. Kang, D. H. Zhai, G. P. Liu, Y. B. Zhao, and P. Zhao, "Stability analysis of a class of hybrid stochastic retarded systems under asynchronous switching," *IEEE Trans. Automatic Control*, vol. 59, no. 6, pp. 1511–1523, 2014.

[4] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability and stabilization of switched linear systems with mode-dependent average dwell time," *IEEE Trans. Automatic Control*, vol. 57, no. 7, pp. 1809–1815, 2012.

[5] Y. Kang, D. H. Zhai, G. P. Liu, and Y. B. Zhao, "On input-to-state stability of switched stochastic nonlinear systems under extended asynchronous switching," *IEEE Trans. Cybernetics*, vol. 46, no. 5, pp. 1092–1105, 2016.

[6] S. Yuan, L. Zhang, B. De Schutter, and S. Baldi, "A novel Lyapunov function for a non-weighted L_2 gain of asynchronously switched linear systems," *Automatica*, vol. 87, pp. 310–317, 2018.

[7] D. Ma and J. Zhao, "Stabilization of networked switched linear systems: an asynchronous switching delay system approach," *Systems and Control Letters*, vol. 77, pp. 46–54, 2015.

[8] J. Lin, S. Fei, and Z. Gao, "Stabilization of discrete-time switched singular time-delay systems under asynchronous switching," *J. the Franklin Institute*, vol. 349, no. 5, pp. 1808–1827, 2012.

[9] Y. E. Wang, J. Zhao, and B. Jiang, "Stabilization of a class of switched linear neutral systems under asynchronous switching," *IEEE Trans. Automatic Control*, vol. 58, no. 8, pp. 2114–2119, 2013.

[10] Z. Xiang, R. Wang, and Q. Chen, "Robust reliable stabilization of stochastic switched nonlinear systems under asynchronous switching," *Applied Mathematics and Computation*, vol. 217, no. 19, pp. 7725–7736, 2011.

[11] G. Zong, R. Wang, W. X. Zheng, and L. Hou, "Finite-time stabilization for a class of switched time-delay systems under asynchronous switching," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 5757–5771, 2013.

[12] H. Liu, Y. Shen, and X. Zhao, "Delay-dependent observer-based H_∞ finite-time control for switched systems with time-varying delay," *Nonlinear Analysis: Hybrid Systems*, vol. 6, no. 3, pp. 885–898, 2012.

[13] J. Lian and Y. Ge, "Robust H_∞ output tracking control for switched systems under asynchronous switching," *Nonlinear Analysis: Hybrid Systems*, vol. 8, pp. 57–68, 2013.

[14] H. Liu, Y. Shen, and X. Zhao, "Asynchronous finite-time H_∞ control for switched linear systems via mode-dependent dynamic state-feedback," *Nonlinear Analysis: Hybrid Systems*, vol. 8, pp. 109–120, 2013.

[15] D. Zhang, L. Yu, Q. G. Wang, and C. J. Ong, "Estimator design for discrete-time switched neural networks with asynchronous switching and time-varying delay," *IEEE Trans. Neural Networks and Learning Systems*, vol. 23, no. 5, pp. 827–834, 2012.

[16] B. Niu and J. Zhao, "Robust h control for a class of switched nonlinear cascade systems via multiple lyapunov functions approach," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6330–6339, 2012.

[17] L. Zhang, S. Zhuang, and P. Shi, "Non-weighted quasi-time-dependent H_∞ filtering for switched linear systems with persistent dwell-time," *Automatica*, vol. 54, pp. 201–209, 2015.

[18] N. Vafamand, M. H. Asemani, and A. Khayatian, "Robust L_1 observerbased non-PDC controller design for persistent bounded disturbed TS fuzzy systems," *IEEE Trans. Fuzzy Systems*, 2017.

[19] J. H. Kim, S. M. Hur, and Y. Oh, "Performance analysis for bounded persistent disturbances in PD/PID-controlled robotic systems with its experimental demonstrations," *Int. J. Control*, vol. 91, no. 3, pp. 688–705, 2018.

[20] Y. Li, Q. Zhang, X. Zhou, and X. Luo, "Basis-dependent and delaydependent results on robust L_1 filtering for networked control systems," *J. the Franklin Institute*, vol. 353, no. 14, pp. 3368–3385, 2016.

[21] L. Zhang, S. Zhuang, P. Shi, and Y. Zhu, "Uniform tube based stabilization of switched linear systems with mode-dependent persistent dwell-time," *IEEE Trans. Automatic Control*, vol. 60, no. 11, pp. 2994–2999, 2015.

[22] L. Zhang, S. Zhuang, and R. D. Braatz, "Switched model predictive control of switched linear systems: feasibility, stability and robustness," *Automatica*, vol. 67, pp. 8–21, 2016.

[23] M. Dehghan and C. J. Ong, "Characterization and computation of disturbance invariant sets for constrained switched linear systems with dwell time restriction," *Automatica*, vol. 48, no. 9, pp. 2175–2181, 2012.

[24] Y. Ren, M. J. Er, and G. Sun, "Switched systems with average dwell time: Computation of the robust positive invariant set," *Automatica*, vol. 85, pp. 306–313, 2017.

[25] C.-K. Lin, "H_∞ reinforcement learning control of robot manipulators using fuzzy wavelet networks," *Fuzzy Sets and Systems*, vol. 160, no. 12, pp. 1765–1786, 2009.

[26] D. Q. Mayne, M. M. Seron, and S. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.

[27] F. Blanchini, D. Casagrande, and S. Miani, "Modal and transition dwell time computation in switching systems: a set-theoretic approach," *Automatica*, vol. 46, no. 9, pp. 1477–1482, 2010.

[28] L. Zhang, N. Cui, M. Liu, and Y. Zhao, "Asynchronous filtering of discrete-time switched linear systems with average dwell time," *IEEE Trans. Circuits and Systems*, vol. 58, no. 5, pp. 1109–1118, 2011.

[29] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*. Nob Hill Pub., 2009.

[30] D. Liberzon, *Switching in Systems and Control*. Springer Science and Business Media, 2012.

[31] S. Kanev and M. Verhaegen, "Controller reconfiguration for non-linear systems," *Control Engineering Practice*, vol. 8, no. 11, pp. 1223–1235, 2000.

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