# Asynchronous Filtering of Discrete-Time Switched Linear Systems With Average Dwell Time

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***Abstract*—Switched dynamical systems can be found in many practical electronic circuits, such as various kinds of power converters, chaos generators, etc. This paper is concerned with the filter design problem for a class of switched system with average dwell time switching. Mode-dependent full-order filters are designed taking a more practical phenomenon, the asynchronous switching into account, where "asynchronous" means that the switching of the filters to be designed has a lag to the switching of the system modes. New results on the stability and $l_2$-gain analyses for the systems are first given where the Lyapunov-like functions during the running time of subsystems are allowed to increase. In light of the proposed Lyapunov-like functions, the desired mode-dependent filters can be designed in that the unmatched filters are allowed to perform in the interval of the asynchronous switching before the matched ones are applied. In $H_\infty$ sense, the problem of asynchronous filtering for the underlying systems in linear cases is formulated and the conditions of the existence of admissible asynchronous filters are obtained. Two examples are provided to show the potential of the developed results.**

***Index Terms*—Asynchronous switching, average dwell time, $H_\infty$ filtering, switched systems.**

## I. Introduction

Switched systems, which are efficiently used to model many physical or man-made systems displaying features of switching, have been extensively studied over the past decades [1], [2]. These kind of dynamical systems exist in a variety of engineering applications, take the electronic circuits field for example, dc/dc convertors [3], oscillators [4], chaos generators [5], to name a few. Typically, switched systems consist of a finite number of subsystems (described by differential or difference equations) and an associated switching signal governing the switching among them. The switching signals may belong to a certain set and the sets may be various. This differentiates switched systems from the general time-varying systems, since the solutions of the former are dependent on both system initial conditions and switching signals [1], [6]. Note that if the switching signal is autonomous and further attached with Markovian stochastic behavior, the resulting system is commonly termed as "Markov jump systems" [7]–[9]. The differences and links between the two categories of hybrid systems have also been investigated, see for example, [7].

The stability problem, caused by diverse switching, is a major concern in the area of switched systems [1], [10]–[17]. To date, two stability issues have been addressed in the literature, i.e., the stability under arbitrary switching and the stability under controlled switching. The former case is mainly investigated by a common Lyapunov function sharing among all the subsystems [1], [18]. An improved method in the discrete-time domain is to adopt the switched Lyapunov function (SLF) proposed in [11]. As for the switched systems under controlled switching, it has been well recognized that the multiple Lyapunov-like function (MLF) approach is more efficient in offering greater freedom for demonstrating stability of the system [10], [12]. Some more general techniques in MLF theory have also been proposed allowing the latent energy function to properly increase even during the running time of certain subsystems except at the switching instants [12], [19]. As a class of typical controlled switching signals, the average dwell time (ADT) switching means that the number of switches in a finite interval is bounded and the average time between the consecutive switching is not less than a constant [1], [20]. The ADT switching can cover the dwell time (DT) switching [1], and its extreme case is actually the arbitrary switching [21]. Therefore, it is of practical and theoretical significance to probe the stability of the switched systems with ADT, and the corresponding results have also been available in [22], [23] for the discrete-time version and [24], [25] for the related applications. Note that, in these results, the Lyapunov-like functions during the running time of subsystems are required to be non-increasing. A recent extension considering partial subsystems to be Hurwitz unstable (the corresponding system energy will be increased) is given in [26] for linear cases in the continuous-time context.

Besides, the $L_2$-gain ("$l_2$" in the discrete-time domain) analysis of switched systems has been frequently related as well [14], [21], [27]–[29]. By the SLF approach, the $l_2$-gain analysis for a class of discrete-time switched systems under arbitrary switching is given in [29]. Imposing different requirements on the used MLF, some results on the $L_2$-gain analysis for the switched systems with DT or ADT switching have also been obtained [21], [28]. Likewise, the considered MLF needs mainly to be non-increasing during the running time of subsystems. In [30], the stability result in [26] was further extended to the $L_2$-gain analysis. A weighted attenuation property is achieved there (i.e., a weighted disturbance attenuation level), and the nonweighted form can be recovered if the weighting is zero, which means that the non-weighted $L_2$-gain of the switched

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systems with ADT is actually bounded by the maximum of all individual $L_2$-gains associated with different subsystems [24]. Note also that the existing results of the $L_2$-gain analysis for the switched systems with ADT are within the continuous-time domain, the discrete-time counterpart has almost not been investigated up to date, with or without considering that the Lyapunov-like functions can be increased.

Moreover, in recent years, state estimation for switched systems has also been widely studied, see for example, [31], [32] and the references therein. A popular solution is to design the less conservative mode-dependent filters and find the admissible switching signals such that the resulting filtering error system is stable and satisfies certain performance. However, a common assumption is that the mode-dependent filters are switched synchronously with the switching of system modes, which is quite ideal. In practice, it inevitably takes some time to identify the system modes and apply the matched filter, thus the asynchronous phenomenon between the system mode switching and the filter switching generally exist[1] . So far, although there are some primary studies on the asynchronous control problems for switched systems, e.g., [33], [34], the asynchronous filtering problem for the systems have not been investigated yet, to the best of our knowledge.

The contribution of this paper lies in that the extended stability and $l_2$-gain results for the switched systems with ADT in the discrete-time nonlinear setting are firstly given by further allowing the Lyapunov-like function to increase during the running time of active subsystems. Then, the asynchronous switching is considered and the $H_\infty$ filtering for the underlying systems in linear cases is studied. The remaining of the paper is organized as follows. In Section II, we review the definitions on stability and $l_2$-gain of switched systems and provide the corresponding results for the switched systems with ADT switching in the discrete-time context. Section III is devoted to derive the results on stability and $l_2$-gain analyses by considering the extended MLF. In Section IV, the conditions of the existence of admissible asynchronous $H_\infty$ filters with the admissible switching are derived in terms of a set of matrix inequalities. Two examples are provided to show the potential and the validity of the obtained results. The paper is concluded in Section 5.

*Notation:* The notation used in this paper is fairly standard. The superscript "$T$" stands for matrix transposition, $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space and $\mathbb{N}$ represents the set of nonnegative integers, the notation $\|\quad\|$ refers to the Euclidean vector norm. $l_2[0,\infty)$ is the space of square summable infinite sequence and for $w=\{w(k)\}\in l_2[0,\infty)$, its norm is given by $\|w\|_2=\sqrt{\sum_{k=0}^{\infty}|w(k)|^2}$. $\mathcal{C}^1$ denotes the space of continuously differentiable functions, and a function $\alpha:[0,\infty)\to[0,\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded, and $\alpha(0)=0$. Also, a function $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t\geq 0$ and $\beta(s,t)$ decreases to 0 as $t\to\infty$ for each fixed $s\geq 0$. Expression $A\Leftrightarrow B$ means $A$ is equivalent to $B$. In addition, in symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for the terms that are introduced by symmetry and $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. The notation $P>0$ $(\geq 0)$ means $P$ is real symmetric and positive definite (semi-positive definite).

[1]In this paper, we slightly abused synchronous (or asynchronous) switching to mean that the switching of system modes and the switching of desired mode-dependent filters are synchronous (respectively, asynchronous). Correspondingly, the delay of asynchronous switching is the time lag from the filters switching to the system modes switching.

## II. Problem Description and Preliminaries

Consider a class of discrete-time switched linear systems given by

$$x(k+1)=A_\sigma x(k)+B_\sigma w(k) \tag{1}$$
$$y(k)=C_\sigma x(k)+D_\sigma w(k) \tag{2}$$
$$z(k)=H_\sigma x(k)+L_\sigma w(k) \tag{3}$$

where $x(k)\in\mathbb{R}^{n_x}$ is the state vector, $w(k)\in\mathbb{R}^{n_w}$ is the disturbance input which belongs to $l_2[0,\infty)$, $z(k)$ is the objective signal to be estimated and $y(k)\in\mathbb{R}^{n_y}$ is the output vector. $\sigma$ is a piecewise constant function of time, called a switching signal, which takes its values in the finite set $\mathcal{I}=\{1,\ldots,N\}$, and $N>1$ is the number of subsystems. At an arbitrary time $k$, $\sigma$ may be dependent on $k$ or $x(k)$, or both, or other logic rules. For a switching sequence $k_0<k_1<k_2<\ldots,\sigma$ is continuous from right everywhere and may be either autonomous or controlled. When $k\in[k_l,k_{l+1})$, we say the $\sigma(k_l)$th subsystem is active and therefore the trajectory $x_k$ of system (1)–(3) is the trajectory of the $\sigma(k_l)$th subsystem. In addition, we exclude Zeno behavior for all types of switching signals as commonly assumed in the literature. The jumps of state for discrete-time system (1)–(3), namely, a continuous signal can not be reconstructed everywhere, is also not considered here.

In this paper, we focus our study of system (1)–(3) on a class of switching signals with ADT switching. The following definitions are recalled.

*Definition 1: [20]:* For switching signal $\sigma$ and any $K>k>k_0$, let $N_\sigma(K,k)$ be the switching numbers of $\sigma$ over the interval $[k,K)$. If for any given $N_0>0$ and $\tau_a>0$, we have $N_\sigma(K,k)\leq N_0+(K-k)/\tau_a$, then $\tau_a$ and $N_0$ are called average dwell time and the chatter bound, respectively.

*Remark 1:* It has been analyzed in [1] that $N_0>1$ gives the switching signals with ADT and $N_0=1$ corresponds exactly to those switching signals with DT. Also, as an extreme case, $\tau_a\to 0$ implies that the constraint on the switching times is almost eliminated and the resulting switching can be arbitrary [21]. Therefore, as a typical set of switching signals with regularities [6], the ADT switching covers both the DT switching and the arbitrary switching and is relatively general.

*Definition 2: [1]:* The switched system (1)–(3) with $w(k)\equiv 0$ is globally uniformly asymptotically stable (GUAS) if there exists a class $\mathcal{KL}$ function $\beta$ such that for all switching signals $\sigma$ and all initial conditions $x(k_0)$, the solutions of (1)–(3) satisfy the inequality $\|x(k)\|\leq\beta(\|x(k_0)\|,k),\forall k\geq k_0$.

*Definition 3:* For $\gamma>0$, system (1)–(3) is said to be GUAS\ with an $l_2$-gain, if under zero initial condition, system (1)–(3) is GUAS and the inequality $\sum_{s=k_0}^{\infty}y^T(s)y(s)\leq\sum_{s=k_0}^{\infty}\gamma^2w^T(s)w(s)$ holds for all nonzero $w(k)\in l_2[0,\infty)$.

Here, we are interested in designing the following mode-dependent full-order filter for system (1)–(3) $\forall \sigma = i \in \mathcal{I}$

$$x_F(k+1) = A_{Fi} x_F(k) + B_{Fi} y(k) \tag{4}$$
$$z_F(k) = C_{Fi} x_F(k) + D_{Fi} y(k) \tag{5}$$

where $A_{Fi}, B_{Fi}, C_{Fi}$ and $D_{Fi}$ are the filter gains to be determined. Also, we aim to consider the more practical asynchronous filtering problem, that is, the switches of the filter gains do not coincide in *real time* with those of system modes. Thus, the resulting filtering error system becomes

$$\begin{cases} \begin{array}{r} \tilde{x}(k+1) = \hat{A}_i \tilde{x}(k) + \hat{E}_i w(k) \\ e(k) = \hat{C}_i \tilde{x}(k) + \hat{F}_i w(k) \end{array}, & \forall k \in [k_l, k_l + \mathcal{T}_{\max}) \\ \begin{array}{r} \tilde{x}(k+1) = A_i \tilde{x}(k) + E_i w(k) \\ e(k) = C_i \tilde{x}(k) + F_i w(k) \end{array}, & \forall k \in [k_l + \mathcal{T}_{\max}, k_{l+1}) \end{cases} \tag{6}$$

where $\tilde{x}(k) = [x^T(k)\ x_F^T(k)]^T$, $e(k) = z(k) \quad z_F(k)$ and

$$\hat{A}_i = \begin{bmatrix} A_i & 0 \\ B_{Fj} C_i & A_{Fj} \end{bmatrix}, \quad \hat{E}_i = \begin{bmatrix} B_i \\ B_{Fj} D_i \end{bmatrix}$$
$$\hat{C}_i = [H_i \quad D_{Fj} C_i \qquad C_{Fj}], \quad \hat{F}_i = L_i \quad D_{Fj} D_i$$
$$A_i = \begin{bmatrix} A_i & 0 \\ B_{Fi} C_i & A_{Fi} \end{bmatrix}, \quad E_i = \begin{bmatrix} B_i \\ B_{Fi} D_i \end{bmatrix}$$
$$\bar{C}_i = [H_i \quad D_{Fi} C_i \qquad C_{Fi}], \quad \bar{F}_i = L_i \quad D_{Fi} D_i.$$

Then, our objective is to design a mode-dependent full-order filter and find a set of admissible switching signals with ADT such that the resulting filtering error systems (6) is GUAS and has a guaranteed $H_\infty$ disturbance attenuation performance, i.e., $\|e\|_2^2 \le \gamma^2 \|w\|_2^2$ for a $\gamma > 0$ in the presence of asynchronous switching.

Before proceeding further, we present the following results on the stability and $l_2$-gain analyses for switched systems in nonlinear setting here for later use.

*Lemma 1: [22]:* Consider switched system $x_{k+1} = f_{\sigma(k)}(x_k)$ and let $0 < \alpha < 1$ and $\mu \ge 1$ be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions $\kappa_1$ and $\kappa_2$ such that $\forall \sigma(k) = i \in \mathcal{I}$, $\kappa_1(\|x_k\|) \le V_i(x_k) \le \kappa_2(\|x_k\|)$, $\Delta V_i(x_k) \triangleq V_i(x_{k+1}) - V_i(x_k) \le -\alpha V_i(x_k)$ and $\forall(\sigma(k_l) = i, \sigma(k_l - 1) = j) \in \mathcal{I} \times \mathcal{I}$, $i \ne j$, $V_i(x_{k_l}) \le \mu V_j(x_{k_l})$, then the system is GUAS for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{\ln \mu}{\ln(1 - \alpha).} \tag{7}$$

*Lemma 2:* Consider switched system $x_{k+1} = f_{\sigma(k)}(x_k, w_k)$, $y_k = h_{\sigma(k)}(x_k)$ and let $0 < \alpha < 1$, $\gamma_i > 0, \forall i \in \mathcal{I}$ be given constants. Suppose that there exist positive definite $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, with $V_{\sigma(k)}(x_{k_0}) \equiv 0$ such that $\forall \sigma(k) = i \in \mathcal{I}, \Delta V_i(x_k) \le -\alpha V_i(x_k) - y_k^T y_k + \gamma_i^2 w_k^T w_k$, then the switched system has a $l_2$-gain no greater than $\gamma = \max\{\gamma_i\}$.

*Remark 2:* Note that the uniformity of stability in Lemmas 1 & 2 means the uniformity over switching signals with the property (7). The proof of Lemma 2 can be completed by referring to the proof of Theorem 2 in [24].

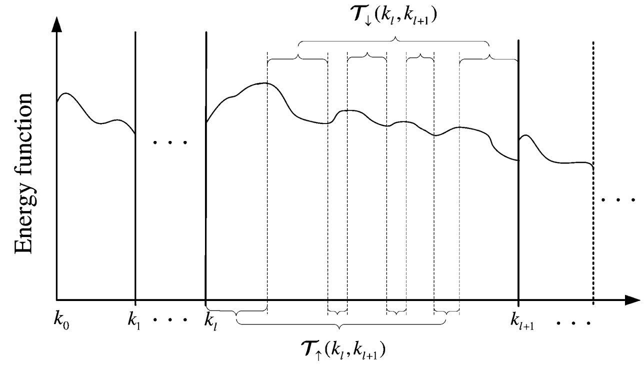


Fig. 1. Extended Lyapunov-like function.

## III. New Stability and $l_2$-Gain Analyses

In this section, by considering a class of Lyapunov-like functions allowed to increase with a bounded increase rate, the improved results for Lemmas 1 and 2 are given, respectively, in order to study the asynchronous filtering problem for system (1) later. For concise notation, let $k_l$ and $k_{l+1}, \forall l \in \mathbb{N}$ denote the starting time and ending time of some active subsystem, while $\mathcal{T}_\uparrow(k_l, k_{l+1})$ and $\mathcal{T}_\downarrow(k_l, k_{l+1})$, imply the total length of the dispersed intervals during which the Lyapunov-like function is increasing and decreasing within the interval $[k_l, k_{l+1})$, respectively. Using $\mathrm{Le}\{[k_l, k_{l+1})\}$ denote the total length of the interval $[k_l, k_{l+1})$, the division gives that $\mathrm{Le}\{[k_l, k_{l+1})\} = \mathcal{T}_\uparrow(k_l, k_{l+1}) + \mathcal{T}_\downarrow(k_l, k_{l+1})$ and Fig. 1 illustrates the considered Lyapunov-like function.

*Lemma 3: (Theorem 1 in [35]):* Consider switched system $x_{k+1} = f_{\sigma(k)}(x_k)$ and let $0 < \alpha < 1$, $\beta \ge 0$ and $\mu \ge 1$ be given constants. Suppose that there exist $\mathcal{C}^1$ functions $V_{\sigma(k)} : \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and two class $\mathcal{K}_\infty$ functions $\kappa_1$ and $\kappa_2$ such that $\forall \sigma(k) = i \in \mathcal{I}$,

$$\kappa_1(\|x_k\|) \le V_i(x_k) \le \kappa_2(\|x_k\|) \tag{8}$$
$$\Delta V_i(x_k) \le \begin{cases} \alpha V_i(x_k), & \forall k \in \mathcal{T}_\downarrow(k_l, k_{l+1}) \\ \beta V_i(x_k), & \forall k \in \mathcal{T}_\uparrow(k_l, k_{l+1}) \end{cases} \tag{9}$$
and $\forall(\sigma(k_l) = i, \sigma(k_l - 1) = j) \in \mathcal{I} \times \mathcal{I}, \quad i \ne j$
$$V_i(x_{k_l}) \le \mu V_j(x_{k_l}) \tag{10}$$

then the system is GUAS for any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{\mathcal{T}_{\max}[\ln(1+\beta) \quad \ln(1 \quad \alpha)] + \ln \mu}{-\ln(1-\alpha)} \tag{11}$$

where $\mathcal{T}_{\max} \triangleq \max_l \mathcal{T}_\uparrow(k_l, k_{l+1}), \forall l \in \mathbb{N}$.

*Remark 3:* Note that the considered energy function in Lemma 3 can be increased both at the switching instants and during the running time of subsystems. However, the possible increment will be compensated by the more specific decrement (by limiting the lower bound of ADT), therefore, the system energy is decreasing from a whole perspective and the system stability is guaranteed accordingly. It is worth noting that an extreme case where $\mathcal{T}_{\max}$ is unbounded is excluded here.

Using the extended Lyapunov-like function as illustrated in Fig. 1, the corresponding $l_2$-gain analysis for system (1)–(3) is given in the following result.

*Lemma 4:* Consider switched system (1)–(3) and let $0 < \alpha < 1$, $\beta \ge 0$ and $\gamma_i > 0, \forall i \in \mathcal{I}$ be given constants. Suppose that there exist positive definite $\mathcal{C}^1$ functions $V_{\sigma(k)}:\mathbb{R}^n\to\mathbb{R}$, $\sigma(k)\in\mathcal{I}$, with $V_{\sigma(k_0)}(x_{k_0})\equiv 0$ such that $\forall(i,j)\in\mathcal{I}\times\mathcal{I}$, $i\neq j$, $V_i(x_{k_l})\le\mu V_j(x_{k_l})$ and $\forall i\in\mathcal{I}$,

$$\Delta V_i(x_k)\le\begin{cases}-\alpha V_i(k)-\Gamma(k), & \forall k\in\mathcal{T}_{\downarrow}(k_l,k_{l+1})\\ \beta V_i(k)-\Gamma(k), & \forall k\in\mathcal{T}_{\uparrow}(k_l,k_{l+1})\end{cases}\tag{12}$$

where $\Gamma(k)\triangleq y_k^Ty_k-\gamma_i^2w_k^Tw_k$, then the switched system is GUAS for any switching signal satisfying (11) and has an $l_2$-gain no greater than $\gamma_s=\max\{\sqrt{\theta^{\mathcal{T}_{\max}-1}}\gamma_i\}$, where $\theta\triangleq(1+\beta)/(1-\alpha)$, $\mathcal{T}_{\max}$ is denoted in (11) and we assume $\mathcal{T}_{\max}\ge 1$.

*Proof:* For $\sigma(k_l)=i\in\mathcal{I}$, $\forall l\in\mathbb{N}$, we say the switched system is active within the $i$th subsystem, then between interval $[k_l,k_{l+1})$, we set the instants $\mathcal{T}_i^1,\mathcal{T}_i^2,\ldots,\mathcal{T}_i^v,\ldots$ as the times when the variation of the Lyapunov-like function changes the direction (from increasing to decreasing or vice versa). Without loss of generality, we assume that the Lyapunov-like function $V_i(x_k)$ is increasing during the interval $[k_l,\mathcal{T}_i^1)$ and decreasing during the interval $[\mathcal{T}_i^m,k)$, $m\in\mathbb{N}$.

Then, for the $i$th subsystem, according to (12), considering $\theta=(1+\beta)/(1-\alpha)$ and denoting $\Gamma(s)\triangleq y_s^Ty_s-\gamma_i^2w_s^Tw_s$ and $\bar\alpha\triangleq 1-\alpha,\tilde\beta\triangleq 1+\beta$, we have

$$\begin{aligned}
&V_i(x_k)\\
&\le\bar\alpha^{k-\mathcal{T}_i^m}V_i(x_{\mathcal{T}_i^m})-\bar\alpha^{k-\mathcal{T}_i^m-1}\Gamma(\mathcal{T}_i^m)-\cdots-\Gamma(k-1)\\
&=\alpha^{k-\mathcal{T}_i^m}V_i(x_{\mathcal{T}_i^m})-\sum_{s=\mathcal{T}_i^m}^{k-1}\alpha^{k-s-1}\Gamma(s)\\
&\le\bar\alpha^{k-\mathcal{T}_i^m}\tilde\beta^{\mathcal{T}_i^m-\mathcal{T}_i^{m-1}}V_i(x_{\mathcal{T}_i^{m-1}})-\sum_{s=\mathcal{T}_i^m}^{k-1}\bar\alpha^{k-s-1}\Gamma(s)\\
&\quad-\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\bar\alpha^{k-\mathcal{T}_i^m}\tilde\beta^{\mathcal{T}_i^m-s-1}\Gamma(s)\le\cdots\\
&\le\alpha^{(k-k_l)}\theta^{\mathcal{T}_{\uparrow}(k-k_l)}V_i(x_{k_l})\\
&\quad-\sum_{s=k_l}^{\mathcal{T}_i^1-1}\bar\alpha^{k-s-1}\theta^{k_l+\mathcal{T}_{\uparrow}(k-k_l)-s-1}\Gamma(s)-\cdots\\
&\quad-\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\alpha^{k-s-1}\theta^{\mathcal{T}_i^m-s-1}\Gamma(s)\\
&\quad-\sum_{s=\mathcal{T}_i^m}^{k-1}\bar\alpha^{k-s-1}\Gamma(s).
\end{aligned}\tag{13}$$

Therefore, under zero condition, one has $V_i(x_{k_l})=0$ and $V_i(x_k)\ge 0$, thus we know that

$$\begin{aligned}
&\sum_{s=k_l}^{\mathcal{T}_i^1-1}\alpha^{k-s-1}\theta^{k_l+\mathcal{T}_{\uparrow}(k-k_l)-s-1}\Gamma(s)+\cdots\\
&\quad+\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\alpha^{k-s-1}\theta^{\mathcal{T}_i^m-s-1}\Gamma(s)\\
&\quad+\sum_{s=\mathcal{T}_i^m}^{k-1}\bar\alpha^{k-s-1}\Gamma(s)\le 0
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{s=k_l}^{\mathcal{T}_i^1-1}\alpha^{k-s-1}\theta^{k_l+\mathcal{T}_{\uparrow}(k-k_l)-s-1}y_s^Ty_s+\cdots\\
&\quad+\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\alpha^{k-s-1}\theta^{\mathcal{T}_i^m-s-1}y_s^Ty_s\\
&\quad+\sum_{s=\mathcal{T}_i^m}^{k-1}\alpha^{k-s-1}y_s^Ty_s\\
&\le\sum_{s=k_l}^{\mathcal{T}_i^1-1}\bar\alpha^{k-s-1}\theta^{k_l+\mathcal{T}_{\uparrow}(k-k_l)-s-1}\gamma_i^2w_s^Tw_s+\cdots\\
&\quad+\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\alpha^{k-s-1}\theta^{\mathcal{T}_i^m-s-1}\gamma_i^2w_s^Tw_s\\
&\quad+\sum_{s=\mathcal{T}_i^m}^{k-1}\bar\alpha^{k-s-1}\gamma_i^2w_s^Tw_s.
\end{aligned}$$

Therefore, from the above and $\mathcal{T}_{\max}\ge 1,\theta>1$,, we can obtain that

$$\begin{aligned}
&\sum_{s=k_l}^{k-1}\alpha^{k-s-1}y_s^Ty_s\\
&=\sum_{s=k_l}^{\mathcal{T}_i^1-1}\bar\alpha^{k-s-1}y_s^Ty_s+\cdots+\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\bar\alpha^{k-s-1}y_s^Ty_s\\
&\quad+\sum_{s=\mathcal{T}_i^m}^{k-1}\alpha^{k-s-1}y_s^Ty_s\\
&\le\sum_{s=k_l}^{\mathcal{T}_i^1-1}\alpha^{k-s-1}\theta^{k_l+\mathcal{T}_{\uparrow}(k-k_l)-s-1}y_s^Ty_s+\cdots\\
&\quad+\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\alpha^{k-s-1}\theta^{\mathcal{T}_i^m-s-1}y_s^Ty_s\\
&\quad+\sum_{s=\mathcal{T}_i^m}^{k-1}\bar\alpha^{k-s-1}y_s^Ty_s\\
&\le\sum_{s=k_l}^{\mathcal{T}_i^1-1}\bar\alpha^{k-s-1}\theta^{k_l+\mathcal{T}_{\uparrow}(k-k_l)-s-1}\gamma_i^2w_s^Tw_s+\cdots\\
&\quad+\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\bar\alpha^{k-s-1}\theta^{\mathcal{T}_i^m-s-1}\gamma_i^2w_s^Tw_s\\
&\quad+\sum_{s=\mathcal{T}_i^m}^{k-1}\alpha^{k-s-1}\gamma_i^2w_s^Tw_s\\
&\le\sum_{s=k_l}^{\mathcal{T}_i^1-1}\alpha^{k-s-1}\theta^{\mathcal{T}_{\max}-1}\gamma_i^2w_s^Tw_s+\cdots\\
&\quad+\sum_{s=\mathcal{T}_i^{m-1}}^{\mathcal{T}_i^m-1}\alpha^{k-s-1}\theta^{\mathcal{T}_{\max}-1}\gamma_i^2w_s^Tw_s
\end{aligned}$$

$$+\sum_{s=\mathcal{T}_i^m}^{k\ \ 1}\alpha^{k-s-1}\theta^{\mathcal{T}_{\max}-1}\gamma_i^2 w_s^T w_s$$
$$=\sum_{s=k_0}^{k\ \ 1}\theta^{\mathcal{T}_{\max}-1}\bar{\alpha}^{k-s-1}\gamma_i^2 w_s^T w_s$$

i.e.,

$$\sum_{s=k_0}^{k\ \ 1}\alpha^{k-s-1}y_s^T y_s \le \sum_{s=k_0}^{k\ \ 1}\theta^{\mathcal{T}_{\max}-1}\alpha^{k-s-1}\gamma_i^2 w_s^T w_s \tag{14}$$

Thus, we have

$$\begin{aligned}&\left[\sum_{k=k_0}^{\infty}\sum_{s=k_0}^{k-1}\alpha^{k\ \ s\ \ 1}y_s^T y_s\right.\\
&\left.\quad\le \sum_{k=k_0}^{\infty}\sum_{s=k_0}^{k\ \ 1}\theta^{\mathcal{T}_{\max}\ \ 1}\alpha^{k\ \ s\ \ 1}\gamma_i^2 w_s^T w_s\right]\\
&\quad\Longleftrightarrow\left[\sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\alpha^{k-s-1}y_s^T y_s\right.\\
&\left.\quad\le \sum_{s=k_0}^{\infty}\sum_{k=s}^{\infty}\theta^{\mathcal{T}_{\max}\ \ 1}\alpha^{k\ \ s\ \ 1}\gamma_i^2 w_s^T w_s\right]\\
&\quad\Longleftrightarrow\left[\sum_{s=k_0}^{\infty}\frac{1}{\alpha\alpha}y_s^T y_s\right.\\
&\left.\quad\le \sum_{s=k_0}^{\infty}\frac{1}{\alpha\alpha}\theta^{\mathcal{T}_{\max}-1}\gamma_i^2 w_s^T w_s\right]\\
&\quad\Longleftrightarrow\left[\sum_{s=k_0}^{\infty}y_s^T y_s \le \sum_{s=k_0}^{\infty}\theta^{\mathcal{T}_{\max}\ \ 1}\gamma_i^2 w_s^T w_s\right].\end{aligned}$$

As a result, for the $i$th subsystem, we know the $l_2$-gain is not greater than $\sqrt{\theta^{\mathcal{T}_{\max}\ \ 1}}\gamma_i$. Therefore, we conclude that system (1)–(3) can have the $l_2$-gain as $\gamma_s=\max\{\sqrt{\theta^{\mathcal{T}_{\max}\ \ 1}}\gamma_i\}$. This completes the proof. □

*Remark 4:* It can be seen that Lemma 3 presents a more general result than Lemma 1 which corresponds to the special case of $\mathcal{T}_{\max}=0$. Note also that if $\mathcal{T}_{\max}=0$, one readily knows from (13) that

$$V_i(x_k)\le \alpha^{k\ \ k_0}V_i(x_{k_0})-\sum_{s=k_0}^{k-1}\alpha^{k\ \ s\ \ 1}\Gamma(s). \tag{15}$$

Then from (15) and the same procedure in the proof for Lemma 4, we can conclude that the switched system is GUAS for any switching signal satisfying (11) and has an $l_2$-gain no greater than $\gamma=\max\{\gamma_i\}$, i.e., Lemma 4 reduces to Lemma 2.

## IV. Asynchronous Filtering

In the results obtained above, a natural question is how $\mathcal{T}_{\max}$ is known in advance. Generally, that is hard since within $[k_l,k_{l+1}),\forall l\in\mathbb{N}^+$, $\mathcal{T}_{\uparrow}(k_l,k_{l+1})$ includes all the randomly dispersed intervals during which the Lyapunov-like function is increasing, consequently, the applications of Lemma 3 and Lemma 4 are actually limited. However, they enable the study on the issues of asynchronous switching, where the corresponding $\mathcal{T}_{\uparrow}(k_l,k_{l+1})$ will be just the interval close to the switching instant as illustrated in Fig. 2. In practice, the interval depends on the identification of system modes and/or the scheduling of the candidate controller/filter gains depending on the control or filtering problems to be solved, then the length of such intervals may vary in different environments. Without loss of generality, we assume that the maximal delay of the asynchronous switching, also denoted by $\mathcal{T}_{\max}$, is known *a priori* here.

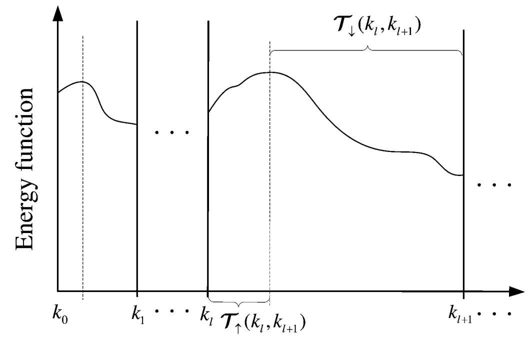


Fig. 2. Typical case of the Extended Lyapunov-like function in Fig. 1.

In this section, we will investigate the filtering problem for the underlying system (1)–(3) in the presence of asynchronous switching. Based on Lemmas 3 and 4, the problem can be solved starting from the so-called bounded real lemma (BRL), which is used to give the stability and $H_\infty$ performance analyses for system (6).

### A. Bounded Real Lemma

By Lemmas 3 and 4, we can obtain a BRL for system (6) as follows.

*Lemma 5:* (Theorem 3 in [35]) Consider the switched linear system (6) and let $0<\alpha<1,\beta\ge 0,\gamma_i>0,\forall i\in\mathcal{I}$ and $\mu\ge 1$ be given constants. If there exist matrices $P_i>0\quad\forall i\in\mathcal{I}$, such that $\forall(i,j)\in\mathcal{I}\times\mathcal{I},i\neq j,P_i\le\mu P_j,\Theta_i\le 0$ and $\Theta_{ij}\le 0$, where

$$\Theta_i\triangleq\begin{bmatrix}-P_i & 0 & P_iA_i & P_iE_i\\ * & -I & C_i & F_i\\ * & * & -(1-\alpha)P_i & 0\\ * & * & * & -\gamma_i^2 I\end{bmatrix} \tag{16}$$
$$\Theta_{ij}\triangleq\begin{bmatrix}-P_i & 0 & P_i\hat{A}_i & P_i\hat{E}_i\\ * & -I & \hat{C}_i & \hat{F}_i\\ * & * & -(1+\beta)P_i & 0\\ * & * & * & -\gamma_i^2 I\end{bmatrix} \tag{17}$$

then under the asynchronous delay $\mathcal{T}_{\max}$, the corresponding system is GUAS for any switching signal satisfying (11) and has a guaranteed $H_\infty$ performance index $\gamma_s=\max\{\sqrt{\theta^{\mathcal{T}_{\max}\ \ 1}}\gamma_i\}$.

### B. $H_\infty$ Filtering

Now a sufficient condition of the existence of the mode-dependent full-order $H_\infty$ filters for the underlying system in the presence of asynchronous switching is given in the following Theorem.

*Theorem 1:* Consider system (1)–(3) and let $0<\alpha<1,\beta\ge 0,\gamma_i>0,\quad\forall i\in\mathcal{I}$ and $\mu\ge 1$ be given constants. If there exist matrices $P_{1i}>0,P_{3i}>0$ and matrices $P_{2i}, X_i, Y_i, Z_i, A_{fi}, B_{fi}, C_{fi}, D_{fi}, \quad \forall i \in \mathcal{I}$ such that $\Phi_i \leq 0, \Phi_{ij} \leq 0$ and

$$\begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix} - \mu \begin{bmatrix} P_{1j} & P_{2j} \\ * & P_{3j} \end{bmatrix} \leq 0 \tag{18}$$

where

$$\Phi_i \triangleq \begin{bmatrix} \Phi_i^{11} & \Phi_i^{12} & 0 & \Phi_i^{14} & X_iB_i + B_{fi}D_i \\ * & \Phi_i^{22} & 0 & \Phi_i^{24} & Z_iB_i + B_{fi}D_i \\ * & * & -I & \Phi_i^{34} & L_i - D_{fi}D_i \\ * & * & * & \Phi_i^{44} & 0 \\ * & * & * & * & \gamma^2 I \end{bmatrix}$$

$$\Phi_{ij} \triangleq \begin{bmatrix} \Phi_{ij}^{11} & \Phi_{ij}^{12} & 0 & \Phi_{ij}^{14} & X_jB_i + B_{fj}D_i \\ * & \Phi_{ij}^{22} & 0 & \Phi_{ij}^{24} & Z_jB_i + B_{fj}D_i \\ * & * & -I & \Phi_{ij}^{34} & L_i - D_{fj}D_i \\ * & * & * & \Phi_{ij}^{44} & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}$$

with $\Phi_i^{11} \triangleq P_{1i} - X_i - X_i^T, \Phi_{ij}^{11} \triangleq P_{1i} - X_j - X_j^T, \Phi_i^{12} \triangleq P_{2i} \quad Y_i \quad Z_i^T, \Phi_{ij}^{12} \triangleq P_{2i} \quad Y_j \quad Z_j^T$ and $\Phi_i^{14} \triangleq [X_iA_i + B_{fi}C_i \; A_{fi}], \Phi_{ij}^{14} \triangleq [X_jA_i + B_{fj}C_i \; A_{fj}], \Phi_i^{24} \triangleq [Z_iA_i + B_{fi}C_i \; A_{fi}], \Phi_{ij}^{24} \triangleq [Z_jA_i + B_{fj}C_i \; A_{fj}], \Phi_i^{34} \triangleq [H_i - D_{fi}C_i - C_{fi}], \Phi_{ij}^{34} \triangleq [H_i - D_{fj}C_i \; - C_{fj}]$

$$\Phi_i^{44} \triangleq \begin{bmatrix} -\alpha P_{1i} & -\alpha P_{2i} \\ * & -\alpha P_{3i} \end{bmatrix}, \Phi_{ij}^{44} \triangleq \begin{bmatrix} -\tilde{\beta} P_{1i} & -\tilde{\beta} P_{2i} \\ * & -\tilde{\beta} P_{3i} \end{bmatrix}$$

$\bar{\alpha} \triangleq 1 - \alpha, \tilde{\beta} \triangleq 1 + \beta$, then there exists a mode-dependent filter with the asynchronous delay $\mathcal{T}_{\max}$ such that the corresponding filtering error system (6) is GUAS for any switching signal with ADT satisfying (11) and has an $H_\infty$ performance index $\gamma_s = \max\{\sqrt{\theta^{\mathcal{T}_{\max}-1}}\gamma_i\}$. Moreover, if feasible solutions exist, the admissible filter gains are given by

$$A_{Fi} = Y_i^{-1}A_{fi}, B_{Fi} = Y_i^{-1}B_{fi}, C_{Fi} = C_{fi}, D_{Fi} = D_{fi}. \tag{19}$$

*Proof:* First of all, for a matrix $R_i, \forall i \in \mathcal{I}$, from the fact $(P_i \quad R_i)^T P_i (P_i \quad R_i) \geq 0$, we have $P_i \quad R_i \quad R_i^T \geq R_i^T P_i \;{}^1 R_i$, then we know the following inequalities

$$\begin{bmatrix} P_i \quad R_i \quad R_i^T & 0 & R_i\bar{A}_i & R_i\bar{E}_i \\ * & I & \bar{C}_i & \bar{F}_i \\ * & * & (1 \quad \alpha)P_i & 0 \\ * & * & * & \gamma_i^2 I \end{bmatrix} \leq 0 \tag{20}$$

$$\begin{bmatrix} P_i - R_j - R_j^T & 0 & R_j\hat{A}_i & R_j\hat{E}_i \\ * & I & \hat{C}_i & \hat{F}_i \\ * & * & (1+\beta)P_i & 0 \\ * & * & * & \gamma_i^2 I \end{bmatrix} \leq 0 \tag{21}$$

guarantee $\Theta_i \leq 0$ and $\Theta_{ij} \leq 0$, respectively (see the proof of [22, Theorem 3] for the similar manipulation). Then, replace $A_i, C_i, E_i, F_i$ and $\hat{A}_i, \hat{C}_i, \hat{E}_i, \hat{F}_i$ in (20) and (21) by the ones in (6) and assume the matrices $P_i, R_i$ to have the following forms:

$$P_i \triangleq \begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix}, \mathrm{R}_i \triangleq \begin{bmatrix} X_i & Y_i \\ Z_i & Y_i \end{bmatrix}.$$

Defining matrix variables

$$A_{fi} = Y_iA_{Fi}, B_{fi} = Y_iB_{Fi}, C_{fi} = C_{Fi}, D_{fi} = D_{Fi} \tag{22}$$

one can readily obtain $\Phi_i$ and $\Phi_{ij}$. Therefore, if $\Phi_i \leq 0, \Phi_{ij} \leq 0$ and (18) holds, we have $\Theta_i \leq 0, \Theta_{ij} \leq 0$ and $P_i \leq \mu P_j$, respectively. According to Lemma 5, the filtering error system (6) is GUAS for any switching signal with ADT satisfying (11) and has an $H_\infty$ performance index $\gamma_s$. In addition, from (22), the mode-dependent filter gains are given by (19). This completes the proof. □

In the absence of asynchronous switching, i.e., $\mathcal{T}_{\max} = 0$ in Theorem 1, we can easily get the following corollary.

*Corollary 1:* Consider switched system (1)–(3) and let $0 < \alpha < 1, \gamma_i > 0, \quad \forall i \in \mathcal{I}$ and $\mu \geq 1$ be given constants. If there exist matrices $P_{1i} > 0, P_{3i} > 0$ and $P_{2i}, X_i, Y_i, Z_i, A_{fi}, B_{fi}, C_{fi}, D_{fi}, \forall i \in \mathcal{I}$ such that $\forall (i,j) \in \mathcal{I} \times \mathcal{I}, i \neq j, \Phi_i \leq 0$ and (18) holds, where $\Phi_i$ is shown in Theorem 1, then there exists a mode-dependent filter such that the resulting filtering error system is GUAS for any switching signal with ADT satisfying (7) and has an $H_\infty$ performance index $\gamma = \max\{\gamma_i\}$. Moreover, if a feasible solution exists, the admissible filter gains are given by (19).

*Remark 5:* Solving the convex problems contained in the above Theorem 1 and Corollary 1, the scalars $\gamma$ and $\gamma_s$ can be optimized in terms of the feasibility of the corresponding conditions. In addition, it is obvious that $\gamma_s \geq \gamma$, which means that the $H_\infty$ performance achieved in the presence of asynchronous switching is worse than the one in the case of synchronous switching. However, the filter designed without considering asynchronous switching, even under the admissible switching (11), may fail to obtain the prescribed (or optimized) $\gamma$ or even $\gamma_s$, which we will show via the example in next subsection.

### C. Examples

In this subsection, we will present two examples to demonstrate the validity of the filter design approach in the presence of asynchronous switching. The first numerical example is used to show the necessity of considering asynchronous switching, and the second example is derived from a PWM-driven boost converter, a typical circuit system to illustrate the applicability of the theoretical results.

*Example 1:* Consider a discrete-time switched linear system (1)–(3) consisting of three subsystems described by

$$A_1 = \begin{bmatrix} -0.60 & -0.05 \\ 0.38 & 0.68 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.63 & 0.23 \\ 0.75 & -0.68 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.75 & -0.15 \\ 0.75 & 0.90 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.30 \\ 0.20 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1.40 \\ 0.30 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.10 \\ 0.10 \end{bmatrix}$$

$$C_1 = [0.10 \quad 0.10], \quad C_2 = [0.30 \quad 0.40]$$

$$C_3 = [\;\; 0.10 \quad 0.20], \quad L_1 = 0.20$$

$$D_1 = 0.40, \quad D_2 = \quad 0.50, \quad D_3 = 0.20$$

$$H_1 = [0.70 \quad 0.30], \quad H_2 = [0.20 \quad 0.40]$$

$$H_3 = [\;\; 0.10 \quad 0.20], \quad L_2 = 0.30, \quad L_3 = \quad 0.10.$$

The maximal delay of asynchronous switching $\mathcal{T}_{\max} = 2$.

The objective is to design a mode-dependent full-order filter and find out the admissible switching signals such that the resulting filtering error system is stable with an optimized $H_\infty$ disturbance attenuation performance.

We shall first demonstrate that if one studies the filtering problem of the above system assuming synchronous switching, i.e., by Corollary 1, the corresponding design results will be invalid in the presence of asynchronous switching. Giving $\mu = 1.05$ and $\alpha = 0.20$ and solving the convex optimization problem in Corollary 1, one can get $\tau_a^* = 0.463, \gamma^* = 0.427$ and the corresponding filter gains as

$$A_{F1} = \begin{bmatrix} 0.48 & 0.12 \\ 0.39 & 0.65 \end{bmatrix}, \quad A_{F2} = \begin{bmatrix} 1.27 & 1.00 \\ 0.32 & 0.18 \end{bmatrix}$$
$$A_{F3} = \begin{bmatrix} 0.25 & 0.24 \\ 0.25 & 0.99 \end{bmatrix}. \tag{23}$$

Due to the space limit, we omit $B_{Fi}, C_{Fi}, D_{Fi}, i = 1,2,3$ here. The filtering error response in Fig. 3(a) shows that the above filter is effective with $\gamma = 0.1536 < 0.4273$ under a switching sequence with $\tau_a = 1 > 0.463$ for given $w(k) = 0.5\exp(\quad 0.05k)$. However, the filtering error responses in the presence of asynchronous switching, plotted in Fig. 3(b)–(d) for the switching sequences with $\tau_a = 1,2,3$, respectively, show that the filtering error system is stable though, the optimized $H_\infty$ performance can not be guaranteed. In other words, the designed filter can not estimate the state of the original system in a required $H_\infty$ performance index. Now, turn to Theorem 1 and consider the asynchronous switching. By further giving $\beta = 0$ and solving the convex optimization problem in Theorem 1, we can get $\tau_a^* = 2.463, \gamma_s^* = 1.872$ and filter gains as ($B_{Fi}, C_{Fi}, D_{Fi}, i = 1,2,3$ are omitted)

$$A_{F1} = \begin{bmatrix} -0.15 & -0.18 \\ 0.43 & 0.52 \end{bmatrix}, \quad A_{F2} = \begin{bmatrix} 0.36 & 0.98 \\ 0.26 & 0.13 \end{bmatrix}$$
$$A_{F3} = \begin{bmatrix} 0.08 & 0.03 \\ 0.60 & 0.58 \end{bmatrix}. \tag{24}$$

Then, for the switching sequences with $\tau_a = 3,4$ (both are greater than 2.463), the filtering error responses using filter (24) are given in Fig. 4(a)–(b). Also, Fig. 5 gives the validation on the $H_\infty$ performance indices that the resulting filter error systems can achieve when applying (23) and (24), respectively, under randomly 200 switching sequences with $\tau_a = 3$. It can be observed from Fig. 3, 4, and 5 that the $H_\infty$ filter (23) designed by Corollary 1 is invalid (even can not ensure $\gamma_s^* = 1.872$), on the contrary, the filter obtained from Theorem 1 is effective in spite of asynchronous switching.

*Example 2:* Consider a PWM (Pulse-Width-Modulation)-driven boost converter, shown in Fig. 6. The switch $s(t)$ is controlled by a PWM device and can switch at most once in each period $T$; $L$ is the inductance, $C$ the capacitance, $R$ the load resistance, and $e_s(t)$ the source voltage. As a typical circuit system, the converter is used to transform the source voltage into a higher voltage. The control problems for such power converters have been widely studied in the literature, such as the optimal control [36], the passivity-based control [37], and the sliding mode control [38], etc. In recent years, the class of power converters is alternatively modeled as switched system and the corresponding stabilization problem has also been investigated [39], [40]. As done in [39], [40], by introducing variables $\tau = t/T, L_1 = L/T$ and $C_1 = C/T$, the differential equations for the boost converter are as follows:

$$\dot{e}_c(\tau) = -\frac{1}{RC_1}e_C(\tau) + (1 - s(\tau))\frac{1}{C_1}i_L(\tau) \tag{25}$$
$$\dot{i}_L(\tau) = \quad (1 \quad s(\tau))\frac{1}{L_1}e_C(\tau) + s(\tau)\frac{1}{L_1}e_S(\tau) \tag{26}$$

Then, (25)–(26) can be further expressed by

$$\dot{x} = A_\sigma^c x, \quad \sigma \in \{1,2\} \tag{27}$$

where $x = [e_C, i_L, 1]^T$ and

$$A_1^c = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{C_1} & 0 \\ -\frac{1}{L_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2^c = \begin{bmatrix} -\frac{1}{RC_1} & 0 & 0 \\ 0 & 0 & \frac{1}{L_1} \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that each mode in (27) is non-Hurwitz and the stabilization problem for it is solved in [40] by designing stabilizing switching laws (the result for the buck-boost converter therein is applicable to the boost converter). As a prerequisite of employing the filtering techniques, however, all the modes of the filtered system (1)–(3) should be stable. Here, differing from [40], we assume that each mode is firstly stabilized by some control law and get a closed-loop continuous-time switched system $\dot{x} = \bar{A}_\sigma^c x, \sigma \in \{1,2\}$, where the two subsystems are both Hurwitz. According to the same normalization technique used in [40], the matrices in (27) can be given by

$$A_1^c = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2^c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the objective in the example is to testify the asynchronous $H_\infty$ filter design techniques and show the potential of the obtained theoretical results in circuit systems, we assume the control matrices for (27) to be $B_1^c = B_2^c = [\quad 0.1 \quad 0.4 \quad 0.5]^T$ and a set of admissible controller gains can be solved as $K_1 = [-6.61 \quad -1.07 \quad -9.32], K_2 = [-5.37 \quad -12.42 \quad -10.07]$. Then, the closed-loop system can be obtained with matrices

$$\bar{A}_1^c = \begin{bmatrix} -0.34 & 1.11 & 0.93 \\ -3.65 & -0.43 & -3.73 \\ -3.30 & -0.54 & -4.66 \end{bmatrix}$$
$$\bar{A}_2^c = \begin{bmatrix} -0.46 & 1.24 & 1.00 \\ -2.15 & -4.97 & -3.03 \\ -2.68 & -6.21 & -5.03 \end{bmatrix}.$$

By setting a certain sampling time $T_s = T/10$ and considering that there exists the disturbance input in the underlying system, one can obtain

$$A_1 = \begin{bmatrix} 0.94 & 0.10 & 0.06 \\ 0.30 & 0.95 & 0.30 \\ 0.25 & 0.06 & 0.63 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0.93 & 0.08 & 0.07 \\ -0.14 & 0.66 & -0.20 \\ -0.16 & -0.40 & 0.66 \end{bmatrix}$$

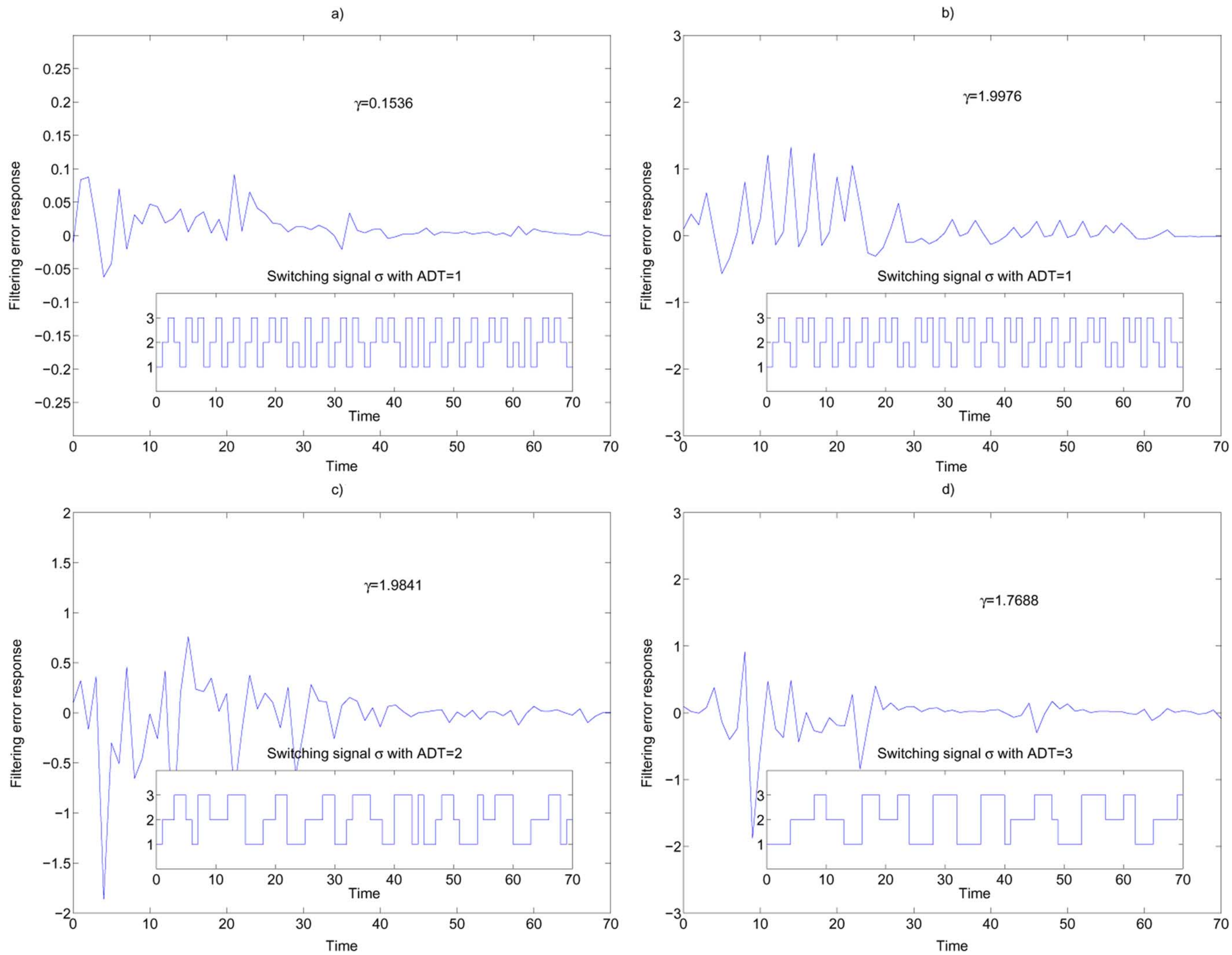


Fig. 3. Filtering error response by filter (23): (a) $T_{\rm Max} = 0, ADT = 1$, (b) $T_{\rm Max} = 2, ADT = 1$, (c) $T_{\rm Max} = 2, ADT = 2$, (d) $T_{\rm Max} = 2, ADT = 3$.

in (1)–(3) and suppose other system matrices to be

$$B_1 = \begin{bmatrix} 0.30 \\ 0.20 \\ 0.10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.40 \\ 0.30 \\ 0.20 \end{bmatrix}$$
$$C_1 = [0.10 \quad 0.10 \quad 0.10], \quad D_1 = 0.4, \quad D_2 = \quad 0.5$$
$$C_2 = [0.30 \quad 0.40 \quad 0.10], \quad L_1 = L_2 = 0$$
$$H_1 = [0.70 \quad 0 \quad 0.30], \quad H_2 = [0.20 \quad 0 \quad 0.40].$$

Also, we assume the maximal delay of asynchronous switching $\mathcal{T}_{\max} = 2$. Then, by giving $\mu = 1.02, \alpha = 0.02, \beta = 0.01$ and solving the convex optimization problem in Theorem 1, we can get $\tau_a^* = 3.9652, \gamma_s^* = 2.2359$ and filter gains as

$$A_{F1} = \begin{bmatrix} 0.84 & 0.02 & 0.23 \\ -0.07 & 0.67 & -0.08 \\ -0.13 & -0.01 & 0.48 \end{bmatrix}$$
$$A_{F2} = \begin{bmatrix} 0.83 & 0.24 & 0.13 \\ -0.37 & 0.96 & -0.38 \\ -0.28 & -0.14 & 0.49 \end{bmatrix}.$$

We also omit $B_{Fi}, C_{Fi}, D_{Fi}, i = 1, 2$ due to space limit. The effectiveness of the desired filter with the above gains can be verified by observing the responses of the filtering error systems in the same rein of Example 1. We only demonstrate the applicability of the developed filter design techniques and omit the curves here.

## V. Conclusions and Future Works

The problems of the stability and $l_2$-gain analyses and $H_\infty$ filtering for a class of discrete-time switched systems with ADT switching are reinvestigated in this paper. By allowing the MLF to increase during the running time of subsystems with a limited increase rate, the more general stability and $l_2$-gain results are obtained. Aiming at a more practical problem that the switching of the filters may have a lag to the switching of system modes, the asynchronous filtering is considered and the existence conditions of the asynchronous $H_\infty$ filters for the underlying systems in linear cases are derived. It is also shown that the obtained conditions cover the cases of synchronous switching. Two examples illustrate the validity and applicability of the obtained theoretical results.

As future works in the theoretical aspect, it is expected that the methodologies behind this paper can be used for the underlying switched systems with parameter perturbations, time delays, etc. It is also worthwhile to investigate the asynchronous

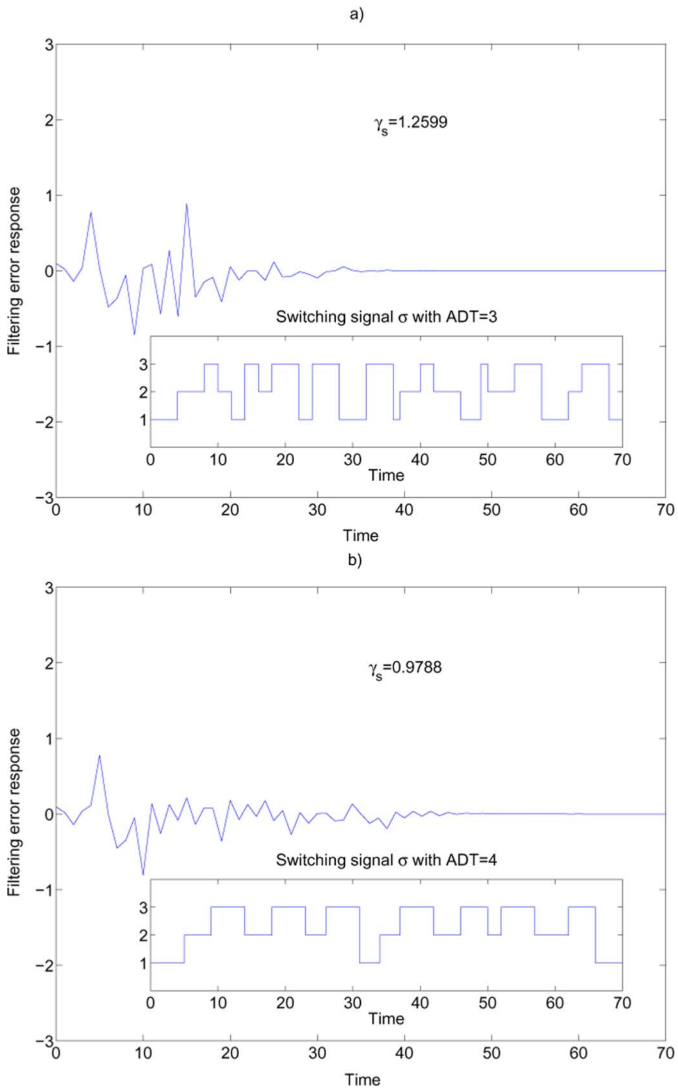


Fig. 4. Filtering error response by filter (24): (a) $T_{\rm Max} = 2, ADT = 3$, (b) $T_{\rm Max} = 2, ADT = 4$.

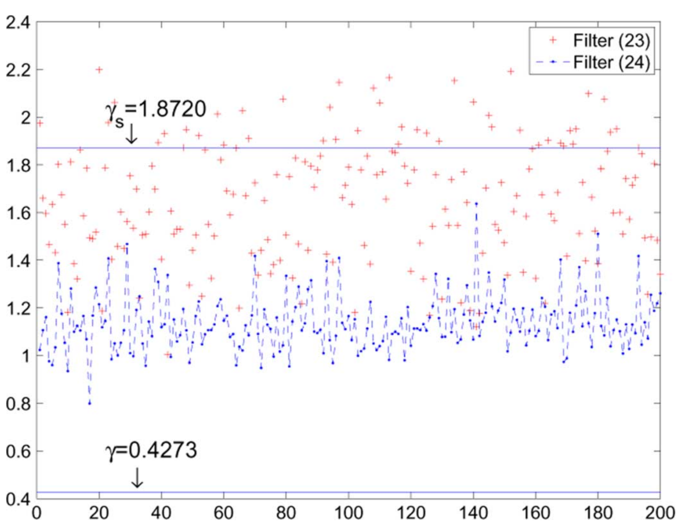


Fig. 5. $H_\infty$ performance indices of filtering error system by filter (23) and filter (24).

switching problems on the switched systems with other classes of switching signals. As for the applications aspect, some examples with higher dimensions, which widely exist in circuits systems field, are significant to be considered and the desired reduced-order filters design need to proceed.

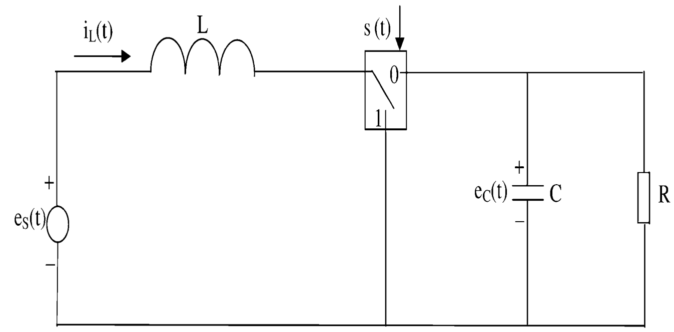


Fig. 6. Boost converter.

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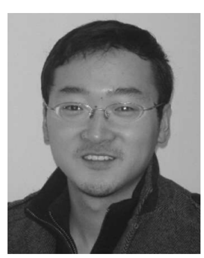

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