Simultaneous Trajectory Optimization and Force Control with Soft Contact Mechanics

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Abstract—Force modulation of robotic manipulators has been extensively studied for several decades but is not yet commonly used in safety-critical applications due to a lack of accurate interaction contact modeling and weak performance guarantees - a large proportion of them concerning the modulation of interaction forces. This study presents a high-level framework for simultaneous trajectory optimization and force control of the interaction between manipulator and soft environments. Sliding friction and normal contact force are taken into account. The dynamics of the soft contact model and the manipulator dynamics are simultaneously incorporated in the trajectory optimizer to generate desired motion and force profiles. A constraint optimization framework based on Differential Dynamic Programming and Alternative Direction Method of Multipliers has been employed to generate optimal control input and high-dimensional state trajectories. Experimental validation of the model performance is conducted on a soft substrate with known material properties using Cartesian space force control mode. Results show a comparison of ground truth and predicted model based contact force states for a few cartesian motions and the validity range of the friction model. Potential applications include high-level task planning of medical tasks involving manipulation of compliant, delicate, and deformable tissues.

I. INTRODUCTION

Robotic applications in medical domain have gained increasing attention over the past few decades [1], [2]. Among this medical domain, control of the interaction forces between a robot and its environment is essential to a variety of safety-critical tasks. For instance, the interaction force ought to be modulated accurately in compliant environments, such as those in a surgical setting, micro-assembly, or biological tissue manipulation. Force control based on identifiable physical models is essential to identify instability modes (e.g., those caused by the bandwidth and structure) and maintain reliable force interaction to guarantee safety. Thus, a model-based trajectory planning method with a high-fidelity contact force model is essential for successful deployment with satisfactory motion tracking and contact force performance.

Compared to rigid contact models, soft contact models are subject to difficulties caused by non-linear material properties and non-uniformity as well as intensive computation burden. Numerous contact models have been presented in the literature to model interactions involving elastic deformation [3]. These models have broad applications and are essential in many engineering areas such as machine design, robotics, multi-body analysis, to name a few. For contact problems that involve elasticity, Hertz adhesive contact theory has been well established [4]. In this study, we focus on robotic tasks interacting with soft tissues, the contact behavior of which is determined by not only external and viscous forces, contact geometry, but also material properties (see Figure 1). The soft contact mechanics are crucial in the physical model identification for applications in surgical robots.

Simultaneous trajectory generation and force control enable sophisticated manipulation tasks in interacting with the complex objects. As a promising approach along this direction, trajectory optimization with contact models has been extensively investigated in the robotics community [5]–[11]. By incorporating the contact dynamics into an optimization problem, contact-dynamics-consistent motions can be generated for complex robot behaviors, such as dynamic locomotion or manipulation. A majority of them focused on rigid contact dynamics [5]–[8]; whereas [9]–[11] directly integrated a soft contact model inside the system dynamics, and implicitly optimized both contact force and other control inputs. In [12], a soft contact model is take into account inside the optimization formulation for Whole-Body Control. However, most of the works above assumed spring-damper...
type soft contact models, which still largely mismatched the contact surface deformation or elasticity in reality. In this study, we leverage a distributed optimization algorithm proposed recently in [13] to solve a constrained trajectory optimization with a high-fidelity soft contact model.

Manipulator contact models are intuitively framed and executed in the task space. In safety-critical tasks such as soft material manipulation and medical applications, force-torque control plays a significant role. These include interaction with humans in proximity or with direct physical contact. In approaching the contact interaction problem, data-driven techniques have been explored to learn the interaction between robotic manipulators and the environment [14], [15]. Unlike rigid contacts, the soft environment is stochastic in nature. Thus, it is challenging to learn the contact model and robot dynamics simultaneously through data. In this study, we present a model for contact interaction and embedded it into the high-level motion planning via enforced constraints.

The main contributions of this study are listed below:

- Presentation of a dynamic interaction model based on soft contact mechanics for a predefined geometry with Hertz visco-static theory.
- Incorporation of the interaction model with a distributed DDP-type trajectory optimization with constraints.
- Experimental validation of the derived contact dynamic model and ground-truth force-torque data.

II. RELATED WORK

Elastic contact mechanics [4] have been extensively studied in various research fields where contact modeling is imperative for safety and performance requirements. Existing works in [16]–[19] have used soft contact models for both analysing and control. These works include quasi-static assumptions and studies of [20], [21] explore cases where high-velocity impacts on soft material are considered. In the impact cases, visco-elastic models have been widely investigated. For instance, studies in [16], [20] compared various visco-elastic models with experimental validations. Overall, a majority of these works have shown that the Hertzian-based hunt-crossey model is the one most suitable for visco-elastic cases. Furthermore, fundamentals of frictional sliding motion are laid in the works of [22], [23] where the main focus is rigid body contacts but could be generalized to soft contacts. [24], [25] are some recent examples where contact-area-based models are proposed.

Trajectory optimization (TO) is a powerful tool to generate reliable and intelligent robot motions. To solve such a TO, various numerical methods have been proposed [26]–[28]. Among them, Differential Dynamic Programming (DDP) and iterative Linear Quadratic Regulator (iLQR) have aroused much attention in solving TO in the context of unconstrained problems, where only dynamics constraint is enforced in the forward-pass. The Riccati-like backward pass in DDP or iLQR effectively reduces the complexity of solving an approximated LQR problem over the whole time horizon, and the optimization is solved in an iterative fashion. In [29], DDP was used in a balancing task of a humanoid robot with high degrees of freedom (DoFs). A more recent work [30] demonstrated a Model Predictive Control (MPC) implementation based on DDP. However, regular DDP algorithms are not capable of addressing constraints. In [31]–[33], DDP-type variants were proposed to cope with state and control constraints. Instead, our approach employs an augmented Lagrangian method named as Alternating Direction Methods of Multipliers (ADMM) [13], [34]–[36] to address various constraints. This ADMM framework is capable of tackling more constraints by introducing additional optimization blocks, making the algorithm suitable for parallel computing.

III. SOFT CONTACT MODELING

A. Contact modeling via Hertz’s theory

In this section, we model the interaction dynamics between an application tool mounted on a manipulator and a soft tissue in terms of contact geometry and mechanics. In the example shown in this study, contact part of manipulation is assumed to be a ball, that is a spherical indentation (for simplicity, but not limited to). Further, we assume that the application tool used is rigid and relatively very stiff compared to the contact surface. Along with these assumptions, we derive the dynamic model based on the contact friction theory and Hertz visco-static theory. According to Hertz’s theory, the largest static indentation is achieved at the central point of the circle (see Figure 2) and can be expressed as:

$$d = \left[ \frac{9F^2}{16E^2R} \right]^{\frac{1}{3}}$$

where $E$ is the reduced Young’s modulus of tool and surface, $R$ is the radius of the tool end, $F$ is the force imparted on the surface by manipulator-end effector. Combined Young’s modulus of the tool and the soft contact surface material can lumped to one term as:

$$\frac{1}{E} = \frac{1 - \nu_2^2}{E_1} + \frac{1 - \nu_2^2}{E_2}$$

where $E_1$, $E_2$ and $\nu_1$, $\nu_2$ are Young’s moduli and Poisson ratios of the end-effector and contact surface material, respectively. In our scenario, we assume the contact part as a rigid object and thus the Young’s modulus of the ball $E_2$ is approximated as infinity. Accordingly, we have $E = E_1/(1 - \nu_1^2)$. The deformation and stress distributions on the surface are approximated by the universal Hooke’s law and Hertz’s theory. Details of normal, radial, and hoop (i.e., moving direction) stress distributions within the contact area in the cylindrical coordinate system are provided in the Appendix.

Accordingly, the deformation distribution is derived from the stress distribution equations as follows:

$$u_z = \begin{cases} \frac{3\pi}{8a} \left[ \frac{1 - \nu^2}{E} \right] m_0 (2a^2 - r^2), & (r \leq a) \\ \frac{3\pi}{8a} \left[ \frac{1 - \nu^2}{E} \right] m_0 \left[ (2a^2 - r^2) \sin^{-1} \left( \frac{r}{a} \right) + a(r^2 - a^2) \right], & (r \geq a) \end{cases}$$
where \( p_m = \frac{F}{\pi a^2} \) is the average stress applied in contact part by manipulation and \( a = \sqrt{Rd} \) is the radius of contact area (see figure 2). The dynamic contact model for a ball (i.e., spherical geometry) is applied with a force vector \( \mathbf{F} \) at an angle \( \theta_F \) to the perpendicular and moves in a circular path of radius \( R \) with a uniform velocity \( \mathbf{v_c} \). This represents the scenario of manipulating an application tool to work with soft tissues. For simplicity, our model focuses on sliding friction with a uniform velocity \( \dot{e} \) and ignores other frictional sources such as adhesion and rolling induced by deformation. Due to the symmetry of our contact scenario, \( \sigma_r \) represents the principal stress within the contact circle. Thus, we can present the stress tensor of any contact point \((r, \theta, z)\) via the Cauchy stress theory [4].

\[
\sigma = \begin{bmatrix}
\sigma_r & 0 & \sigma_{rz} \\
0 & \sigma_\theta & 0 \\
\sigma_{rz} & 0 & \sigma_z 
\end{bmatrix}
\]  

(2)

Since the task is defined in the Cartesian frame, we convert parameters to Cartesian coordinates from cylindrical coordinates. The stress tensor in Cartesian coordinate is \( \sigma_e = T^T \sigma T \), where the transformation matrix \( T \) is defined in Appendix.

At an arbitrary point on contact surface \((x, y, z)\), the normal vector from this point to centroid of ball is \( \mathbf{n} = [\cos \theta \ 0 \ \sin \theta]^T \). The normal stress of the contact surface is \( \sigma_n = \mathbf{n}^T \sigma \mathbf{n} \) with

\[
\sigma_n = \sigma_r c^2 \theta^2 + \sigma_\theta s^4 \theta + \sigma_z c^2 \theta + 2 \sigma r s \theta c^2 \theta
\]

Given this stress expression, the overall friction force of the contact surface is represented as

\[
df = \mu \sigma_n dS = \mu \sigma_n \times 2 \pi r \frac{dr}{\cos \theta}
\]

\[
f = \int df \cos \theta = 2 \pi \mu \int_0^z \sigma_n r dr
\]

(3)

(4)

In the surface normal direction, it is assumed that the surface is in contact with the end point of the tool. As a result, Eq. (1) always holds. The derivative form of Eq. (1) is

\[
\dot{z} = -\dot{d} = -\frac{1}{6E^2 RF_z} \dot{F}_z
\]

(5)

where \( z \) represents the position along the surface normal direction of the contact point and force along the normal direction is defined as \( F_z = F \cos \theta_F \). In the moving direction, by using Eq. (4) and using \( \mu F_z \) to get the frictional force caused by the normal force \( \mu F_z \), total friction is:

\[
F_\theta = f = \mu F_z \left[ 1 + (2 \nu - 1) \frac{3a^2}{10R^2} \right] + k_d \dot{e}
\]

(6)

where \( k_d \) is a damping coefficient in the moving direction. By substituting \( a = \sqrt{Rd} \) and Eq. (6), we have the derivative form of Eq. (6)

\[
\dot{F}_\theta = \dot{f} = \mu \dot{F}_z + \frac{3 \mu(2 \nu - 1)}{10 R} \left( \dot{F}_z d - F_z \dot{z} \right) + k_d \ddot{e}
\]

(7)

In the radial direction, we have \( c = m\dot{v}/R_c \), where \( m \) is the mass of the tool and \( R_c \) is the radius of the curvature path. Then the derivative of \( F_r \) is

\[
\dot{F}_r = \frac{2m\dot{v}}{R_c}
\]

(8)

B. Other visco-elasticity contact models

Apart from the Hertz model that we used, there exist other contact models based on visco-elasticity [16], [20] that are proven to effectively estimate the interaction between the soft material and tools. The well-received models are:

1) Kelvin–Voigt (KV) Model: model consists of a spring in parallel with a damper.

\[
F_e = K \delta x + D \delta \dot{x}
\]

2) Hunt and Crossey Model: This model is a modification to the Hertz quasi-static model with a non-linear damping term.

\[
F_e = K \delta x^n + \frac{\lambda \delta x \delta \dot{x} \dot{x}}{A_2}
\]

where, \( \delta x \) is the indentation depth, \( K \) and \( D \) are stiffness and damping coefficients, and \( F_e \) is the contact force associated with it along the surface normal.

Visco-elastic models yield better results when there exist hard impacts or high velocities involved in the direction of penetration. This is due the presence of \( A_1 \) and \( A_2 \) damping terms in the model. [20] did a quantitative study on a comparison of different visco-elastic models. This is in-fact useful in force modulation in non-stationary environments where penetration depth changes frequently. We limit our scope to the Hertz quasi-static model in our formulation, but it could be extended to the models mentioned above.

IV. MANIPULATION DYNAMICS AND TOOL USE MODEL

A. Manipulator and tool model

We use the dynamics of the manipulator as well as the dynamics of the tool in the optimization formalization along with the contact dynamical model.

We use the wrench control capability of the manipulator as described in IV-E. In general, the dynamics of a manipulator model can be expressed as:

\[
\ddot{q} = M(q)^{-1}(\tau_u - C(q, \dot{q})q - G(q) - J^T w_u)
\]

(9)
where \( \mathbf{q} \) is the joint state vector, \( \mathbf{M}(\mathbf{q}) \) is the joint space mass matrix, \( \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \) is the Coriolis term, \( \mathbf{G}(\mathbf{q}) \) is the gravity term, \( \tau \) is the torque applied at joints, \( \mathbf{J} \) is the end-effector Jacobian and \( \mathbf{w}_u \) is the external Cartesian force at the end-effector.

**Remark 1:** In this study, manipulator and tool dynamics are modeled separately. The underlying motivation is driven by our experimental design procedure and is due to the fact that the manipulator model is not very well-identified.

### B. Application tool model

We model the application tool as a rigid body with a contact point and an wrench input \( \mathbf{w}_u \) as shown in Figure 3. This is independent from the manipulator model in the previous section. The Cartesian position of the contact point is defined as \( \mathbf{x}_c = [x_c \ y_c \ z_c] \). Then, the tool dynamics can be represented as:

\[
\begin{align*}
\dot{\mathbf{x}}_c &= \mathbf{H}^{-1} (\mathbf{w}_u + \mathbf{m} \mathbf{g} - \mathbf{F}_c) \\
\end{align*}
\]  

(10)

where \( \mathbf{H} = \text{diag}[m, I_{xx}, I_{yy}, I_{zz}] \in \mathbb{R}^{6 \times 6} \) is the mass of the tool and \( I_{xx}, I_{yy}, I_{zz} \) are the moments of inertia around center of mass. \( \dot{\mathbf{w}}_c \) and \( \dot{\mathbf{\omega}}_c \) are the centroidal dynamics of the tool while \( \mathbf{F}_c \) is the force at the contact point of the tool.

\[
\begin{align*}
\dot{\mathbf{w}}_c &= \mathbf{\omega}_c, \quad \dot{\mathbf{\omega}}_c = \dot{\mathbf{\omega}}_c \\
[\ddot{\psi}_c \ \ddot{\phi}_c] &\quad T = \mathbf{T}_c^{-1} [\dot{\mathbf{\omega}}_c - \dot{\mathbf{T}}_c [\dot{\psi}_c \ \dot{\phi}_c] T]
\end{align*}
\]

where \( \mathbf{\omega}_{c,e} = \mathbf{T} \dot{\mathbf{\psi}}_c \ddot{\phi}_c \) is the angular rate of the tool, \( \dot{\mathbf{\omega}}_{c,e} = \mathbf{T} \dot{\mathbf{\psi}}_c \ddot{\phi}_c + \mathbf{T} \dddot{\mathbf{\psi}}_c \dot{\phi}_c \dddot{\phi}_c \). \( \mathbf{T} \) is the corresponding mapping from the Euler rate [38] to the angular rate. Tool centroid dynamics and end-point dynamics are related by

\[
\begin{align*}
\dot{\mathbf{x}}_c &= \mathbf{x}_c + \mathbf{\omega}_c \times \mathbf{r}_{ce} \\
\dot{\mathbf{x}}_e &= \mathbf{x}_c + \mathbf{\omega}_c \times \mathbf{r}_{ce} + \mathbf{\omega}_c \times (\mathbf{\omega}_c \times \mathbf{r}_{ce})
\end{align*}
\]

### C. Contact model

Let the moving direction (unit vector) of the tool be \( \mathbf{n}_u \) and the direction orthogonal to the moving direction as \( \mathbf{n}_\perp \).

**Remark 2:** see euler error section (chapter 2) of [37]

Then,

\[
\begin{align*}
\mathbf{F}_c &= \left( (6 E^2 R F_z)^{1/3} \right) \mathbf{n}_z \\
&+ \left( \mu R F_z + 3 \mu (2 \nu - 1) \right) \left( \mathbf{F}_z d + \mathbf{F}_z d \right) \mathbf{n}_v + \frac{2 m v^2}{R} \mathbf{n}_v 
\end{align*}
\]

(11)

where \( d \) is the deformation at central point of contact circle and calculated from Eq. (1), and \( \mathbf{F}_z \) is the vertical force (the surface normal direction) applied on the surface by the manipulator and \( v = ||\dot{\mathbf{x}}_e|| \) is the moving velocity of the tool contact point.

### D. Trajectory optimization formulation

The state for our trajectory optimization is represented as: \( \mathbf{x}_E = [x_e \ y_e \ z_e \ \psi_e \ \phi_e] \), \( \mathbf{x}_M = [q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6 \ q_7] \), \( \mathbf{x} = [\mathbf{x}_M \ \mathbf{x}_E] \), \( \mathbf{F}_c \in \mathbb{R}^{4 \times 4} \), \( \mathbf{u} = [\mathbf{w}_u \ \mathbf{\tau}_u] \) Orientation is represented by Euler angles.

**Overall optimization problem is formulated as:**

\[
\begin{align*}
\min_{\phi} & \; \sum_{i=0}^{N} \delta \mathbf{x}[i]^T \mathbf{Q} \delta \mathbf{x}[i] + \mathbf{u}[i]^T \mathbf{P} \mathbf{u}[i] \\
&+ \mathbf{F} \mathbf{K}(\delta \mathbf{x}_M[i])^T \mathbf{P} \mathbf{F} \mathbf{K}(\delta \mathbf{x}_M[i]) \\
\text{s.t.} & \; \mathbf{x}[i+1] = \mathbf{F}(\mathbf{x}[i], \mathbf{u}[i]) \\
& \; \mathbf{x}[0] = \mathbf{x}_0 \\
& \; \mathbf{X}_M \leq \mathbf{x}_M \leq \mathbf{X}_M \\
& \; \mathbf{w}_u \leq \mathbf{w}_u \leq \mathbf{w}_u \\
& \; \frac{mv^2}{R} \leq \mu N^T \mathbf{F}_c \mathbf{N} + \mathbf{G}(\mathbf{x}_e) 
\end{align*}
\]

(12a)

(12b)

(12c)

(12d)

(12e)

where \( \delta \mathbf{x}[i] = (x[i] - x[i]) \) is the current state error with respect to reference \( x[i] \), \( \mathbf{Q} \in \mathbb{R}^{n \times n} \) and \( \mathbf{P} \in \mathbb{R}^{n \times m} \) are the state and control weighting matrices, \( \mathbf{F} \mathbf{K} \in \mathbb{R}^{4 \times 4} \) is the forward kinematics function for the manipulator and \( \mathbf{P} \) is the state weighting matrix for the forward function. For simplicity, we use \( \phi = (x[0], \ldots, N), \mathbf{u}[0], \ldots, N - 1) \) to represent the sequence of state-control pairs. \( \mathbf{N}[i] = \mathbf{N} \) is the surface average normal vector (as precept by the force-torque sensor). In this study, we use \( \mathbf{N} = [0 \ 0 \ 1]^T \), which is only in the \( z \) direction. \( v = \dot{x}_e \) and \( \dot{R} \) is the curvature (see figure 3) of the radius of the path tracked and \( \mathbf{G} \) is a general term to compensate for added deformation (rolling) friction.

### E. Manipulator wrench control

The manipulator used in our experiment was a 7-DOF KUKA LBR R820. Inertial parameters for this manipulator are not accurately identified to the precision of modulating small scale magnitudes as used in our experiments. As an alternative, KUKA FRI (Fast Research Interface [39]) allows users to command external wrenches at the end effector accurately. It uses the proprietary internal dynamic model of the manipulator to generate commanded external wrenches.
wrenches. Furthermore, this allows us to define desired Cartesian impedance for the manipulator. The commanded wrench gives an input to the base of the application tool. The manipulator control law is expressed as:

\[ \mathbf{w}_u = \mathbf{w}_u + K \left( \mathbf{x}_E^T[c] - \mathbf{x}_E^T[d] \right) \tag{13} \]

where \( K \) stands for the user specified proportional gain, i.e., stiffness in Cartesian space. \( \mathbf{x}_E^T[c] - \mathbf{x}_E^T[d] \) is the distance between the current pose and the desired pose of end-effector (with the tool part). In our optimal control problem (12), we take this external wrench into consideration as part of the control variables.

V. TRAJECTORY OPTIMIZATION WITH CONTACT DYNAMICS

Differential Dynamic Programming (DDP) is well known for effectively solving unconstrained trajectory optimization. Generally speaking, DDP is an indirect method which only optimizes control, and the dynamics constraint is automatically satisfied during the forward pass. Given an initial guess of control inputs, an updated state trajectory is generated by forward propagating the differential dynamic equation. Then a quadratic approximation is constructed to approximate the cost function and dynamics around the current trajectory so that a Riccati recursion can be used to derive the optimal feedback control law efficiently. By iteratively updating the state and control trajectories, the solution will converge.

Based on the manipulator and tool models in Sec. IV-A and the proposed contact dynamics in Sec. IV-C, DDP is used to generate desired joint and Cartesian motion and force profiles with dynamic constraints.

One limitation of DDP is to address constraints. Since our contact model enforces frictional constraints, it is desired to incorporate these contact constraints along with the state and control constraints. The work in [13] uses an iterative process based on Alternating Direction Method of Multipliers (ADMM) to solve for constraints, in particular for robots with rigid body dynamics.

The ADMM algorithm decomposes a large-scale, holistic optimization problem into sub-problems and solves the optimization iteratively. In each iteration, the primal and dual variables are updated sequentially. Under mild conditions, both primal and dual variables converge to the optimal solutions. Readers are referred to [40] for more details about ADMM formulations. To make this algorithm applicable to our case, we transcribe the original optimal control problem (12) into the following form:

\[
\begin{align*}
\min_{\phi, \mathbf{u}, \mathbf{x}} & \sum_{i=0}^{N} C(\mathbf{x}[i], \mathbf{u}[i]) + I_D(\mathbf{x}[i], \mathbf{u}[i]) \\
& + I_{\mathcal{J}, \mathcal{U}, \mathcal{F}}(\mathbf{x}_M[i], \mathbf{u}[i], \mathbf{\lambda}[i]) \\
\text{s.t.} & \mathbf{x}_M = \bar{\mathbf{x}}_M, \quad \mathbf{u} = \bar{\mathbf{u}}, \quad \mathbf{\lambda} = \bar{\mathbf{\lambda}}
\end{align*}
\tag{14}
\]

where we define \( C(\mathbf{x}, \mathbf{u}) \) as the local cost function corresponding to Eq. (12a) and \( \mathbf{\lambda} = (\mathbf{x}_M^T, \mathbf{F}_e^T, \mathbf{\lambda}_e^T)^T \). We use \( \phi = (\mathbf{x}_M[0, \ldots, N], \mathbf{u}[0, \ldots, N-1], \mathbf{\lambda}[0, \ldots, N]) \) to express the concatenated states and controls that are required to be projected. The admissible set \( \mathcal{D} \) represents the generalized dynamics constraint (12b), where \( \mathcal{D} = \{ (\mathbf{x}, \mathbf{u}) \mid \mathbf{x}_0 = \mathbf{x}_{\text{init}}, \mathbf{x}_{i+1} = \mathcal{F}(\mathbf{x}_i, \mathbf{u}_i), i = 0, 1, \ldots, N-1 \} \). The closed and convex sets \( \mathcal{J}, \mathcal{U} \) and \( \mathcal{F} \) stand for joint limit (12d), control limit (12e) and contact constraint (12f), respectively.

In general, an indicator function regarding a closed convex residual (see [40], Sec. 3.3) is satisfied.

Then for each ADMM iteration \( k \), the updating sequence in a scaled form is

\[
\phi_{k+1} = \arg \min_{\phi} C(\mathbf{x}, \mathbf{u}) + I_D(\mathbf{x}, \mathbf{u}) + \frac{\rho_u}{2} \| \mathbf{x}_M - \bar{\mathbf{x}}_M^k + \mathbf{v}_u^k \|_2^2 \\
+ \frac{\rho_u}{2} \| \mathbf{u} - \bar{\mathbf{u}}^k + \mathbf{v}_u^k \|_2^2 + \frac{\rho_f}{2} \| \mathbf{\lambda} - \bar{\mathbf{\lambda}}^k + \mathbf{v}_f^k \|_2^2
\tag{16a}
\]

\[
\bar{\phi}_{k+1} = \arg \min_{\phi} I_{\mathcal{J}, \mathcal{U}, \mathcal{F}}(\mathbf{x}_M, \mathbf{u}, \mathbf{\lambda}) + \frac{\rho_u}{2} \| \mathbf{x}_M^k - \bar{\mathbf{x}}_M + \mathbf{v}_u^k \|_2^2 \\
+ \frac{\rho_u}{2} \| \mathbf{u}^k - \bar{\mathbf{u}} + \mathbf{v}_u^k \|_2^2 + \frac{\rho_f}{2} \| \mathbf{\lambda}_e^k - \bar{\mathbf{\lambda}} + \mathbf{v}_f^k \|_2^2
\tag{16b}
\]

where \( \phi \) and \( \bar{\phi} \) are primal variables. \( \mathbf{v}_j, \mathbf{v}_u, \mathbf{v}_f \) are dual variables related to joint limits, control limits and contact constraints. \( \rho_j, \rho_u, \rho_f \) are step-size parameters corresponding to each constraint.

Note that for Eq. (16a), we use DDP to solve it and \( I_D \) is always zero since the state trajectory is always dynamically feasible by performing the forward pass. For Eq. (16b), this minimization problem reduces to a projection operator on convex sets \( \mathcal{J}, \mathcal{U}, \) and \( \mathcal{F} \)

\[
\bar{\phi}_{k+1} = \arg \min_{\phi \in \mathcal{C}} \frac{\rho_j}{2} \| \mathbf{x}_M^k - \bar{\mathbf{x}}_M + \mathbf{v}_j^k \|_2^2 \\
+ \frac{\rho_u}{2} \| \mathbf{u}^k - \bar{\mathbf{u}} + \mathbf{v}_u^k \|_2^2 + \frac{\rho_f}{2} \| \mathbf{\lambda}_e^k - \bar{\mathbf{\lambda}} + \mathbf{v}_f^k \|_2^2
\]

\[
\mathcal{C} = \{ (\mathbf{x}_M, \mathbf{u}, \mathbf{\lambda}) \mid \mathbf{x}_M \in \mathcal{J}, \mathbf{u} \in \mathcal{U}, \mathbf{\lambda} \in \mathcal{F} \}
\]

We use a saturation function to efficiently project the infeasible values onto the boundaries induced by different constraints. The whole process of our ADMM algorithm is shown in Algorithm 1. The selection of \( \bar{\phi} \) and dual variables \( \mathbf{v} \) are arbitrary, and we initialize them as zero. The initial trajectory of \( \phi \) is generated by running forward pass with an initial guess of controls. In each ADMM iteration, the controls from last ADMM iteration will be sent to the current DDP solver as a warm-start, which makes the DDP solver converge faster within around ten iterations in each ADMM iteration after the initial one. Then the trajectories are solved iteratively until a stopping criterion with regard to primal residuals (see [40], Sec. 3.3) is satisfied.
on a material testing platform INSTRON®. The young modulus was estimated through a non-linear least square estimator by using Eq. (1). Eq. (6) was used to estimate the frictional coefficients. Frictional force magnitude in the moving direction $F_{\text{fric}}$, velocity magnitude $V_e$, and normal contact force $F_z$ were calculated from the collected data and a three dimensional robust least square approximation was done (with a logistic distance function in MATLAB©). Robust least square fitting was used to mitigate the sensitivity to the model deviation as the deformation increases (see Fig. 5). Identified Young Modulus and friction coefficient were incorporated into the overall optimization in Eq. (12). Desired states to track were the desired end-effector position $(x_e, y_e, z_e)$ and the desired contact force $F_z$. Optimal trajectories and inputs (Cartesian force $w_u$) were generated through the optimization problem stated in the formulation of Eq. (12). Constraints were satisfied within $10^{-2}$ residual value violations in both primal and dual stopping criteria. Desired contact force was maintained within the bounded valid range of the friction model.

The identification of material surface properties is two-fold. First, to validate that the used models are well suited and to use in the trajectory optimization framework to generate optimal open-loop trajectories. Friction data fitting results are presented in Figure 5. Data were fit with an resulting R-squared value of 0.9103. Frictional coefficient ($\mu = 0.4512$) and damping coefficient ($k_d = 13.1315$) were identified.

It is observed that with the increase of normal force on the surface, the effects of deformation dominates the frictional forces. This phenomenon is due to the increased rolling friction and material-specific artifacts, e.g., non-uniformity in frictional coefficient and stress distribution. Moreover, the presence of fluids or any micro-particular particles will increase the non-uniformity.

**B. Motion along a desired path with desired contact force**

Open-loop control trajectories generated from our trajectory optimization were input to the manipulator (cartesian wrench as described in section IV-E). We generated trajectories along a line, a circle, and number eight shape. Figure 6 shows the results where the first row depicts motion tracking while the second row shows the ground truth and actual values of $F_x, F_y, F_z$. Since the control was in wrench mode, motion and forces are coupled. Any mismatch in required contact forces (e.g., those due to friction and deformation) would directly affect the motion and vice versa. Although the tracking performance is not superior but adequate due to the unmodelled dynamics, stiction, and non-uniformity of the soft material.

**VII. Conclusion and Future Work**

In automation tasks where biological or soft tissue contact is required, it is paramount to design soft contact interaction models where controllers can be designed to guarantee safety performance. Proper identification of the contact material and the ability to perform mundane tasks such as incisions

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**Algorithm 1 ADMM trajectory optimization**

1. $\phi \leftarrow \phi^0, \dot{\phi} \leftarrow \dot{\phi}^0$
2. $v_j \leftarrow v_j^0, v_u \leftarrow v_u^0, v_f \leftarrow v_f^0$
3. repeat
4. $\phi \leftarrow \text{DDP} (\phi, \dot{\phi} - v_j, \ddot{u} - v_u, \ddot{\lambda} - v_f)$
5. $\phi \leftarrow \text{Projection} (\phi + v_j, u + v_u, \lambda + v_f)$
6. $v_j \leftarrow v_j + x - \dot{x}$
7. $v_u \leftarrow v_u + u - \ddot{u}$
8. $v_f \leftarrow v_f + \lambda - \ddot{\lambda}$
9. until stopping criterion is satisfied
10. return $\phi$

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![Experimental Setup](image.png)

Fig. 4: Experimental Setup. The application tool with a spherical geometry in contact with a soft material.

![Friction model validation and identification](image.png)

Fig. 5: Friction model validation and identification. $\mu = 0.4512, K_d = 13.1315$, R-squared = 0.9103.
along given paths, and motion disturbance compensation are the examples where contact modelling are crucial. This study presents a coherent framework to perform simultaneous motion and force modulation on compliant surfaces. Potential applications of this work include contact manipulation in soft tissues or safety-critical environments.

In summary, we derived a soft contact dynamical model and incorporated it into a trajectory optimizer that is capable of handling state, control and contact constraints. Trajectories solved from the trajectory optimizer were experimentally validated on a soft surface (EcoFlex®) with the aid of a robot manipulator with an attached spherical shaped tool tip. Surface material properties were estimated and further used in generating optimal trajectories. Ground truth forces were obtained using a force-torque sensor (ATI mini45) and compared against the predicted results. Results show accurate tracking of the forces and desired positions as predicted by the derived dynamic contact model.

Future work will include real-time implementation of the trajectory optimization and execution in a model predictive control (MPC) fashion. Implementing a low-level adaptive controller to handle inherent stochasticity will be given equal priority.

APPENDIX

In this Appendix, we provide the details on several stress distributions. First, the normal Stress Distribution $\sigma_z$:

$$\frac{\sigma_z}{p_m} = -\frac{3}{2} \left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}} \quad (r \leq a)$$  \hspace{1cm} (17)

Radical Stress Distribution $\sigma_r$:

$$\frac{\sigma_r}{p_m} = \frac{2\nu - 1}{2} \frac{a^2}{r^2} \left[1 - \left(1 - \frac{r^2}{a^2}\right)^n\right] - 3\nu \left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}} \quad (r \leq a)$$

Hoop Stress Distribution $\sigma_\theta$:

$$\frac{\sigma_\theta}{p_m} = \frac{1 - 2\nu}{2} \frac{a^2}{r^2} \left[1 - \left(1 - \frac{r^2}{a^2}\right)^n\right] - \frac{3}{2} \left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}} \quad (r \leq a)$$

where $p_m = F/(\pi a^2)$ is the average stress applied in contact part by manipulation and $a = \sqrt{Rd}$ is the radius of contact area (refer to figure 2). The transformation matrix $T$ is

$$T = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

REFERENCES


